Structural decompositions in module theory and their constraints

9th International Algebraic Conference in Ukraine
Lviv, July 8, 2013

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Overview

Part I: Decomposable classes
Part I: Decomposable classes (the rare jewels)
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Classic decomposition theorems.
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- **Part I: Decomposable classes** *(the rare jewels)*
  - Classic decomposition theorems.

- **Part II: Deconstructible classes**
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  - Filtrations and transfinite extensions.
  - Deconstructibility and approximations.
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- **Part I: Decomposable classes** *(the rare jewels)*
  1. Classic decomposition theorems.

- **Part II: Deconstructible classes** *(the ubiquitous mainstream)*
  1. Filtrations and transfinite extensions.
  2. Deconstructibility and approximations.

- **Part III: Non-deconstructibility**
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- **Part III: Non-deconstructibility** (reaching the limits)
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  2. Deconstructibility and approximations.

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Part I: Decomposable classes
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(blocks put in a row)
Definition

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[Gruson-Jensen’73], [Huisgen-Zimmermann’79]

**Mod-$R$ is decomposable, iff $R$ is right pure-semisimple.**

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uniqueness by Krull-Schmidt-Azumaya.
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$\text{Mod-} R$ is decomposable, iff $R$ is right pure-semisimple.

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E.g., $I \oplus I^{-1} \cong R^{(2)}$ for each non-principal ideal $I$ of a Dedekind domain $R$. 

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Definition

A class of modules \( C \) is decomposable, provided there is a cardinal \( \kappa \) such that each module in \( C \) is a direct sum of \( < \kappa \)-generated modules from \( C \).

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[Faith-Walker’67]  
**The class** \( \mathcal{I}_0 \) **of all injective modules is decomposable, iff \( R \) is right noetherian.**
Here, \( \kappa \) depends \( R \); uniqueness by Krull-Schmidt-Azumaya.
Part II: Deconstructible classes
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(blocks put on top of other blocks)
Definition

Let \( C \subseteq \text{Mod-}R \). A module \( M \) is \( C \)-filtered (or a transfinite extension of the modules in \( C \)), provided that there exists an increasing sequence \( (M_\alpha \mid \alpha \leq \sigma) \) consisting of submodules of \( M \) such that \( M_0 = 0, M_\sigma = M \),

- \( M_\alpha = \bigcup_{\beta < \alpha} M_\beta \) for each limit ordinal \( \alpha \leq \sigma \), and
- for each \( \alpha < \sigma \), \( M_{\alpha+1}/M_\alpha \) is isomorphic to an element of \( C \).
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**Notation:** \( M \in \text{Filt}(C) \).

A class \( \mathcal{A} \) is **closed** under transfinite extensions, if \( \text{Filt}(\mathcal{A}) \subseteq \mathcal{A} \).
**Definition**

Let $\mathcal{C} \subseteq \text{Mod-} R$. A module $M$ is $\mathcal{C}$-filtered (or a transfinite extension of the modules in $\mathcal{C}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of $M$ such that $M_0 = 0$, $M_\sigma = M$,

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A class $\mathcal{A}$ is **closed** under transfinite extensions, if $\text{Filt}(\mathcal{A}) \subseteq \mathcal{A}$.

**Eklof Lemma**

The class $\perp \mathcal{C} := \text{KerExt}^1_R (-, \mathcal{C})$ is closed under transfinite extensions for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$. 
The ubiquity of deconstructible classes
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Definition (Eklof’06)

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal $\kappa$ such that $\mathcal{A} \subseteq \text{Filt}(\mathcal{A}^{<\kappa})$, where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$. 

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The ubiquity of deconstructible classes

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### [Enochs et al.’01]

For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.
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[Eklof-T.’01], [Štovíček-T.’09]
For each set of modules \( \mathcal{S} \), the class \( \perp(\mathcal{S}^{\perp}) \) is deconstructible. Here, \( \mathcal{S}^{\perp} := \ker \text{Ext}^1_R(S, -) \).
Approximations of modules
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A class of modules $\mathcal{A}$ is **precovering** if for each module $M$ there is $f \in \text{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \text{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through $f$:

\[ \begin{array}{ccc}
A & \xrightarrow{f} & M \\
\uparrow & & \downarrow \\
A' & \xrightarrow{f'} & M
\end{array} \]

The map $f$ is called an $\mathcal{A}$–precover of $M$. 
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[Saorín-Šťovíček’11], [Enochs’12]

All deconstructible classes closed under transfinite extensions are precovering.

In particular, so are the classes $\perp(S\perp)$ for all sets of modules $S$. 
Some questions
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Is each class of modules closed under transfinite extensions deconstructible/precovering?
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What about the classes of the form \( \perp C \)?
Part III: Non-deconstructible classes
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(no block pattern at all)
First examples
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[Eklof-Shelah’03]

Let \( \mathcal{W} := \perp \{\mathbb{Z}\} \) denote the class of all Whitehead groups. It is independent of ZFC whether \( \mathcal{W} \) is precovering (or deconstructible).
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Let $W := \perp \{\mathbb{Z}\}$ denote the class of all Whitehead groups. It is independent of ZFC whether $W$ is precovering (or deconstructible).

A result in ZFC

A module $M$ is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left $R$-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.
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Assume that $R$ is not right perfect.

- [Herbera-T.’12] The class $\mathcal{FM}$ of all flat Mittag-Leffler modules is closed under transfinite extensions, but it is not deconstructible.
- [Šaroch-T.’12], [Bazzoni-Štovíček’12] If $R$ is countable, then $\mathcal{FM}$ is not precovering.
Further questions
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Is non-deconstructibility a more general phenomenon?
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Still open

Can the class $\bot C$ be non-deconstructible/non-precovering in ZFC?
Locally $\mathcal{F}$-free modules
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Let $R$ be a ring, and $\mathcal{F}$ a class of countably presented modules.
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**Definition**

A module $M$ is **locally $\mathcal{F}$-free**, if $M$ possesses a subset $S$ consisting of countably $\mathcal{F}$-filtered modules, such that

- each countable subset of $M$ is contained in an element of $S$,
- $0 \in S$, and $S$ is closed under unions of countable chains.
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Let $\mathcal{L}$ denote the class of all locally $\mathcal{F}$-free modules.
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**Note:** If $M$ is countably generated, then $M$ is locally $\mathcal{F}$-free, iff $M$ is countably $\mathcal{F}$-filtered.
Flat Mittag-Leffler modules are locally $\mathcal{F}$-free
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**Theorem (Herbera-T.’12)**

Let $\mathcal{F}$ be the class of all countably presented projective modules. Then the notions of a locally $\mathcal{F}$-free module and a flat Mittag-Leffler module coincide for any ring $R$. 

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Flat Mittag-Leffler modules are locally $\mathcal{F}$-free

Theorem (Herbera-T.’12)

Let $\mathcal{F}$ be the class of all countably presented projective modules. Then the notions of a locally $\mathcal{F}$-free module and a flat Mittag-Leffler module coincide for any ring $R$.

For instance, if $R = \mathbb{Z}$, then an abelian group $A$ is flat Mittag-Leffler, iff all countable subgroups of $A$ are free.

In particular, the Baer-Specker group $\mathbb{Z}^\kappa$ is flat Mittag-Leffler for each $\kappa$, but not free.
Trees for locally $\mathcal{F}$-free modules
Let $\kappa$ be an infinite cardinal, and $T_\kappa$ be the set of all finite sequences of ordinals $< \kappa$, so

$$T_\kappa = \{ \tau : n \to \kappa \mid n < \omega \}.$$
Trees for locally $\mathcal{F}$-free modules

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Partially ordered by inclusion, $T_\kappa$ is a tree, called the tree on $\kappa$. 
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Let $\text{Br}(T_\kappa)$ denote the set of all branches of $T_\kappa$. Each $\nu \in \text{Br}(T_\kappa)$ can be identified with an $\omega$-sequence of ordinals $< \kappa$:

$$\text{Br}(T_\kappa) = \{ \nu : \omega \to \kappa \}.$$
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$$\text{Br}(T_\kappa) = \{ \nu : \omega \rightarrow \kappa \}.$$ 

$\text{card } T_\kappa = \kappa$ and $\text{card } \text{Br}(T_\kappa) = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of $\tau$ for each $\tau \in T_\kappa$. 
The Bass modules
Let $R$ be a ring, and $\mathcal{F}$ be a class of countably presented modules.
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$$F_0 \xrightarrow{g_0} F_1 \xrightarrow{g_1} \ldots \xrightarrow{g_{i-1}} F_i \xrightarrow{g_i} F_{i+1} \xrightarrow{g_{i+1}} \ldots$$

with $F_i \in \mathcal{F}$ and $g_i \in \text{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$. 

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with $F_i \in \mathcal{F}$ and $g_i \in \text{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

**Example**

Let $\mathcal{F}$ be the class of all countably generated projective modules. Then the Bass modules coincide with the countably presented flat modules.
Decorating trees by Bass modules
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Let $D := \bigoplus_{\tau \in T_\kappa} F_\ell(\tau)$, and $P := \prod_{\tau \in T_\kappa} F_\ell(\tau)$. 
Decorating trees by Bass modules

Let $D := \bigoplus_{\tau \in T_\kappa} F_{\ell(\tau)}$, and $P := \prod_{\tau \in T_\kappa} F_{\ell(\tau)}$.

For $\nu \in \text{Br}(T_\kappa)$, $i < \omega$, and $x \in F_i$, we define $x_{\nu i} \in P$ by

$$\pi_{\nu}|_i(x_{\nu i}) = x,$$

$$\pi_{\nu}|_j(x_{\nu i}) = g_{j-1} \ldots g_i(x) \text{ for each } i < j < \omega,$$

$$\pi_{\tau}(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_{\tau} \in \text{Hom}_R(P, F_{\ell(\tau)})$ denotes the $\tau$th projection for each $\tau \in T_\kappa$. 

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Constraints for structural decompositions
Decorating trees by Bass modules

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$$\pi_{\nu \upharpoonright j}(x_{\nu i}) = g_{j-1} \ldots g_i(x) \text{ for each } i < j < \omega,$$

$$\pi_{\tau}(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_{\tau} \in \text{Hom}_R(P, F_{\ell(\tau)})$ denotes the $\tau$th projection for each $\tau \in T_\kappa$.

Let $X_{\nu i} := \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of $P$ isomorphic to $F_i$. 
The locally $\mathcal{F}$-free module $L$
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Let $X_\nu := \sum_{i<\omega} X_{\nu i}$, and $L := \sum_{\nu \in \text{Br}(T_\kappa)} X_\nu$. 
The locally $\mathcal{F}$-free module $L$

Let $X_\nu := \sum_{i<\omega} X_{\nu_i}$, and $L := \sum_{\nu \in \text{Br}(T_\kappa)} X_\nu$.

Lemma

- $D \subseteq L \subseteq P$.
- $L/D \cong N(\text{Br}(T_\kappa))$.
- $L$ is locally $\mathcal{F}$-free.
The locally $\mathcal{F}$-free module $L$

Let $X_{\nu} := \sum_{i<\omega} X_{\nu i}$, and $L := \sum_{\nu \in \text{Br}(T_\kappa)} X_{\nu}$.

**Lemma**
- $D \subseteq L \subseteq P$.
- $L/D \cong N(\text{Br}(T_\kappa))$.
- $L$ is locally $\mathcal{F}$-free.

**Lemma (Slávik-T.)**
- $\mathcal{L}$ is closed under transfinite extensions.
- $\mathcal{L}^\perp \subseteq (\lim_{\omega} \mathcal{F})^\perp$. 

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Non-deconstructibility of locally $\mathcal{F}$-free modules
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- $\mathcal{F}$ a class of countably presented modules,
- $\mathcal{L}$ the class of all locally $\mathcal{F}$-free modules,
- $\mathcal{D}$ the class of all direct summands of the modules $M$ that fit into an exact sequence
  \[ 0 \to F' \to M \to C' \to 0, \]
where $F'$ is a free module, and $C'$ is countably $\mathcal{F}$-filtered.
Non-deconstructibility of locally $\mathcal{F}$-free modules

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**Theorem (Slávik-T.)**

Assume there exists a Bass module $N \notin \mathcal{D}$. Then the class $\mathcal{L}$ is not deconstructible.
Flat Mittag-Leffler modules revisited
Corollary

$\mathcal{FM}$ is not deconstructible for each non-right perfect ring $R$. 
Corollary

\( \mathcal{FM} \) is not deconstructible for each non-right perfect ring \( R \).

Proof: If \( R \) a non-right perfect ring, then there is a strictly decreasing chain of principal left ideals

\[
R a_0 \supsetneq \cdots \supsetneq R a_n \cdots a_0 \supsetneq R a_{n+1} a_n \cdots a_0 \supsetneq \ldots
\]
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\]

Let \( \mathcal{F} \) be the class of all countably presented projective modules. Consider the Bass module \( N \) which is a direct limit of the chain

\[
R \to^{a_0} R \to^{a_1} \cdots \to^{a_{i-1}} R \to^{a_i} R \to^{a_{i+1}} \cdots
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Flat Mittag-Leffler modules revisited

Corollary

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\]

Let \(F\) be the class of all countably presented projective modules. Consider the Bass module \(N\) which is a direct limit of the chain

\[
R \xrightarrow{a_0} R \xrightarrow{a_1} \cdots \xrightarrow{a_{i-1}} R \xrightarrow{a_i} R \xrightarrow{a_{i+1}} \cdots
\]

Then there is a non-split pure-exact sequence

\[
0 \rightarrow R^{(\omega)} \xrightarrow{f} R^{(\omega)} \rightarrow N \rightarrow 0,
\]

where \(f(1_i) = 1_i - a_i.1_{i+1}\) for all \(i < \omega\). Then \(N \notin D = P_0\).
Infinite dimensional tilting modules
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Definition

$T$ is a tilting module provided that

- $T$ has finite projective dimension,
- $\text{Ext}_R^i(T, T^{(\kappa)}) = 0$ for each cardinal $\kappa$, and
- there exists an exact sequence $0 \rightarrow R \rightarrow T_0 \rightarrow \cdots \rightarrow T_r \rightarrow 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \text{Add}(T)$, i.e., $T_i$ is a direct summand of a (possibly infinite) direct sum of copies of $T$.\[\text{Add}(T)\]
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$\mathcal{B}_T := \{ T \}^{\perp_\infty} = \bigcap_{1 < i} \text{KerExt}^i_R(T, -)$ the right tilting class of $T$. 

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- there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \text{Add}(T)$, i.e., $T_i$ is a direct summand of a (possibly infinite) direct sum of copies of $T$.

$\mathcal{B}_T := \{ T \}^{\perp_\infty} = \bigcap_{1<i} \ker \text{Ext}^i_R(T, -)$ the right tilting class of $T$.

$\mathcal{A}_T := \perp \mathcal{B}_T$ the left tilting class of $T$. 
Some infinite dimensional tilting theory
Some infinite dimensional tilting theory

Theorem (A model-theoretic characterization of right tilting classes)

Tilting classes are exactly the classes of finite type, i.e., the classes of the form $S^\perp$, where $S$ is a set of strongly finitely presented modules of bounded projective dimension.
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Let $S_T := \mathcal{A}_T \cap \text{mod-}R$ and $\bar{\mathcal{A}}_T := \lim S$. Then $\mathcal{A}_T$ is the class of all direct summands of $S_T$-filtered modules, and $\mathcal{A}_T \subseteq \bar{\mathcal{A}}_T$. 
Some infinite dimensional tilting theory

**Theorem (A model-theoretic characterization of right tilting classes)**

Tilting classes are exactly the classes of finite type, i.e., the classes of the form $S^\perp$, where $S$ is a set of strongly finitely presented modules of bounded projective dimension.

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**Definition**

The tilting module $T$ is $\Sigma$-pure split provided that $\bar{A}_T = A_T$, that is, the left tilting class of $T$ is closed under direct limits. Equivalently: Each pure embedding $T_0 \hookrightarrow T_1$ such that $T_0, T_1 \in \text{Add}(T)$ splits.
Some infinite dimensional tilting theory

**Theorem (A model-theoretic characterization of right tilting classes)**

*Tilting classes are exactly the classes of finite type, i.e., the classes of the form $S^\perp$, where $S$ is a set of strongly finitely presented modules of bounded projective dimension.*

Let $S_T := \mathcal{A}_T \cap \text{mod-}R$ and $\bar{\mathcal{A}}_T := \lim_{\longrightarrow} S$. Then $\mathcal{A}_T$ is the class of all direct summands of $S_T$-filtered modules, and $\mathcal{A}_T \subseteq \bar{\mathcal{A}}_T$.

**Definition**

The tilting module $T$ is $\sum$-pure split provided that $\bar{\mathcal{A}}_T = \mathcal{A}_T$, that is, the left tilting class of $T$ is closed under direct limits. Equivalently: Each pure embedding $T_0 \hookrightarrow T_1$ such that $T_0, T_1 \in \text{Add}(T)$ splits.

**Example (Bass)**

Let $T = R$. Then $T$ is a tilting module of projective dimension 0, and $T$ is $\sum$-pure split, iff $R$ is a right perfect ring.
Locally free modules and tilting
Locally free modules and tilting

The setting

Let $R$ be a countable ring, and $T$ be a non-$\sum$-pure-split tilting module. Let $\mathcal{F}_T$ be the class of all countably presented modules from $\mathcal{A}_T$, and $\mathcal{L}_T$ the class of all locally $\mathcal{F}_T$-free modules.
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Theorem (Slávik-T.)

Assume that $\mathcal{L}_T \subseteq \mathcal{P}_1$, $\mathcal{L}_T$ is closed under direct summands, and $M \in \mathcal{L}_T$ whenever $M \subseteq L \in \mathcal{L}_T$ and $L/M \in \mathcal{A}_T$. Then the class $\mathcal{L}_T$ is not precovering.
Locally free modules and tilting

The setting
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Corollary
If $R$ is countable and non-right perfect, then $\mathcal{FM}$ is not precovering.
Finite dimensional hereditary algebras
Finite dimensional hereditary algebras

Let $R$ be an indecomposable hereditary artin algebra of infinite representation type, with the Auslander-Reiten translation $\tau$. Then there is a partition of all indecomposable finitely generated modules into three sets:

$q :=$ indecomposable preinjective modules
(i.e., indecomposable injectives and their $\tau$-shifts),

$p :=$ indecomposable preprojective modules
(i.e., indecomposable projectives and their $\tau^{-}$-shifts),

$t :=$ regular modules (the rest).
The Lukas tilting module and the Baer modules
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$p^\perp$ is a right tilting class.
$M \in p^\perp$ iff $M$ has no non-zero direct summands from $p$. 
The Lukas tilting module and the Baer modules

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The tilting module $L$ inducing $p^\perp$ is called the Lukas tilting module. The left tilting class of $L$ is the class of all Baer modules.
\( p^\perp \) is a right tilting class.
\( M \in p^\perp \) iff \( M \) has no non-zero direct summands from \( p \).

The tilting module \( L \) inducing \( p^\perp \) is called the Lukas tilting module. The left tilting class of \( L \) is the class of all Baer modules.

\[ \text{[Angeleri-Kerner-T.'10]} \]

The class of all Baer modules coincides with \( \text{Filt}(p) \).

The Lukas tilting module \( L \) is countably generated, but has no finite dimensional direct summands, and it is not \( \sum \)-pure split.
Non-deconstructibility in the realm of artin algebras
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Let $F_L$ be the class of all countably presented Baer modules. The elements of $L_L$ are called the locally Baer modules.
Non-deconstructibility in the realm of artin algebras

Let $\mathcal{F}_L$ be the class of all countably presented Baer modules. The elements of $\mathcal{L}_L$ are called the locally Baer modules.

**Theorem (Slávik-T.)**

Let $R$ be a countable indecomposable hereditary artin algebra of infinite representation type. Then the class $\mathcal{L}_L$ is not precovering (and hence not deconstructible).
A conjecture
A conjecture

A ring \( R \) is right pure-semisimple, iff each class of right \( R \)-modules closed under transfinite extensions and direct summands is deconstructible.