Closure properties of lim $\mathcal{C}$

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In honor of Lidia Angeleri Hügel

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L.Positelski, J.Trlifaj: *Closure properties of* $\varinjlim \mathcal{C}$, arXiv:2110.13105v1, 52 pp.

The class $\varinjlim \mathcal{C}$ - the case of small modules

Let R be a ring, and $\mathcal{C}$ be a class of (right R-) modules closed under finite direct sums.

Denote by $\varinjlim \mathcal{C}$ the class of all modules that are direct limits of direct systems consisting of modules from $\mathcal{C}$.

Lenzing'83

Assume $\mathcal{C}$ consists of finitely presented modules. Then $M \in \varinjlim \mathcal{C}$, iff each homomorphism from a finitely presented module to M factorizes through a module in $\mathcal{C}$.

The class $\varinjlim \mathcal{C}$ is closed under direct sums, pure submodules, pure extensions, and pure epimorphic images. In particular, $\varinjlim \mathcal{C}$ is closed under direct limits, and $\varinjlim \mathcal{C}$ is a covering class.

The class $\varinjlim \mathcal{C}$ - the case of small modules

Angeleri-T.'04

Assume $\mathcal{C}$ that consists of FP_2 modules, $R \in \mathcal{C}$, and $\mathcal{C}$ is closed under extensions and direct summands. Let $\mathcal{L} = \varinjlim \mathcal{C}$. Then $\mathcal{L} = {}^{\intercal}(\mathcal{C}^{\intercal})$.

Hence $\mathcal{L}$ is a covering class closed under transfinite extensions (i.e., $\mathrm{Filt}(\mathcal{L}) \subseteq \mathcal{L}$), and $\mathcal{L}$ is κ^+-deconstructible for $\kappa = \mathrm{card}\ R + \aleph_0$, i.e., $\mathcal{L} \subseteq \mathrm{Filt}(\mathcal{L}^{\leq \kappa})$. So $\mathcal{L} = \mathrm{Filt}(\mathcal{L}^{\leq \kappa})$.

The class $\varinjlim \mathcal{C}$ - the general case

Proposition

The class $\varinjlim \mathcal{C}$ is always closed under direct sums.

But $\varinjlim \mathcal{C}$ need not be closed under direct summands or pure extensions. In particular, $\varinjlim \mathcal{C}$ need not be closed under direct limits:

Examples

- Let R be a commutative von Neumann regular semiartinian ring, and $\mathcal{C}$ the class of all finitely generated completely reducible modules. Then $\mathcal{C}$ is closed under (pure) extensions, but $\varinjlim \mathcal{C}$ is not.
- [Angeleri-T.'04] Let $R = \mathbb{Z}$ and $M \in \mathrm{Mod}\text{–}\mathbb{Z}$ be torsion-free, rigid (i.e. $\mathrm{End}(M) = \mathbb{Z}$), and of rank $r > 1$. Then $\varinjlim \mathrm{add}(M)$ is not closed under direct summands.

The structure of $\varinjlim \mathbf{add}(M)$

Theorem

Let M be a module and $S = \mathrm{End}(M)$. Denote by $\mathcal{F}_S$ the class of all flat right S-modules. Then

$$\varinjlim \mathrm{add}(M) = \{F \otimes_S M \mid F \in \mathcal{F}_S\}.$$

Proof:

1. $\varinjlim \mathrm{add}(M) = \varinjlim \mathrm{sum}(M)$.
2. $\mathrm{Hom}_R(M^m, M^n) \cong M_{n \times m}(S)$.
3. $N = \varinjlim_{i \in I} M^{n_i}$, iff $N \cong F \otimes_S M$, where $F \in \varinjlim_{i \in I} S^{n_i}$, that is, $F \in \mathcal{F}_S$. □

The structure of $\varinjlim \text{add}(M)$

Corollary

The class $\varinjlim \text{add}(M)$ is deconstructible for each module $M \in \mathrm{Mod}\text{–}R$.

However, the class $\varinjlim \text{add}(M)$ need not be closed under extensions even if $\text{add}(M)$ is:

Example

Let $R = \mathbb{Z}_p$ and $M = \mathbb{J}_p$ for a prime p. Then $\text{add}(M) = \text{sum}(M)$ is closed under extensions in $\mathrm{Mod}\text{–}R$.

The class $\varinjlim \text{add}(M)$ (= the class of all torsion-free $\mathbb{J}_p$-modules, but viewed as $\mathbb{Z}_p$-modules) is not extension closed in $\mathrm{Mod}\text{–}R$.

The structure of $\varinjlim \mathbf{add}(M)$

The old example revisited

Let $R = \mathbb{Z}$ and M be a torsion-free rigid abelian group of rank $r > 1$.

Then all non-zero groups in $\varinjlim \operatorname{add}(M)$ are of the form $F \otimes_S M$ for $0 \neq F \in \mathcal{F}_R$, so they are torsion-free of rank ≥ 2. Hence $\mathbb{Q} \notin \varinjlim \operatorname{add}(M)$.

However, $E(M) \cong \mathbb{Q}^{(r)}$ is a direct limit of a suitable countable direct system of the form $M \to M \to \dots$, whence $\mathbb{Q}^{(r)} \in \varinjlim \operatorname{add}(M)$.

So in general, $\varinjlim \operatorname{add}(M)$ need not be closed under extensions or direct limits.

The structure of $\varinjlim \mathbf{Add}(M)$

Theorem

Let M be a module and $\mathfrak{S} = \mathrm{End}(M)$ be its endomorphism ring endowed with the finite topology (whose base of neighborhoods of zero is formed by the annihilators of finitely generated submodules of M). Then M is a discrete left $\mathfrak{S}$-module.

Denote by $\mathcal{F}_{\mathfrak{S}}$ the class of all right $\mathfrak{S}$-contramodules that are direct limits of direct systems of projective right $\mathfrak{S}$-contramodules. Then

$$\varinjlim \mathrm{Add}(M) = \{F \odot_{\mathfrak{S}} M \mid F \in \mathcal{F}_{\mathfrak{S}}\}$$

where $F \odot_{\mathfrak{S}} M$ denotes the contratensor product of the right $\mathfrak{S}$-contramodule F with the discrete left $\mathfrak{S}$-module M.

Corollary

The class $\varinjlim \mathrm{Add}(M)$ is deconstructible for each module $M \in \mathrm{Mod}\text{–}R$.

Sketch of the proof

1. There is a natural equivalence between the additive categories $\mathrm{Add}(M)$ and $\mathcal{P}_{\mathfrak{S}}$, where the latter denotes the category of all projective right $\mathfrak{S}$-contramodules ($=$ direct summands of free right $\mathfrak{S}$-contramodules).

2. The equivalence above extends to an adjunction (Ψ_M, Φ_M) beween $\mathrm{Mod}\text{–}R$ and $\mathrm{Contra}\text{–}\mathfrak{S}$, where the latter denotes the category of all right $\mathfrak{S}$-contramodules. Here, $\Psi_M = \mathrm{Hom}_R(M, -)$ and $\Phi_M = - \odot_{\mathfrak{S}} M$.

3. Being a left adjoint, Φ_M preserves direct limits. The proof then proceeds as in the case of $\varinjlim \mathrm{add}(M)$. □

Another description of $\varinjlim \mathcal{C}$

Lemma

Let $\mathcal{C}$ be a class of modules closed under arbitrary direct sums. Then $\varinjlim \mathcal{C}$ coincides with the class of all modules M of the form $M = C/K$ where

- $C \in \mathcal{C}$,
- $K \subseteq C$, and K is a directed union of a direct system of its submodules, K_i $(i \in I)$, such that
- K_i is a direct summand in C, and $C/K_i \in \mathcal{C}$, for each $i \in I$.

Easy proof: Let K be the kernel of the canonical presentation of the direct limit as a factor of the direct sum. Then K has the form described above. □

In particular, each $M \in \varinjlim \mathcal{C}$ is a direct limit of a direct system consisting of modules from $\mathcal{C}$, and of split epimorphisms.

$\varinjlim \mathbf{add}(M)$ versus $\varinjlim \mathbf{Add}(M)$

For any class of modules $\mathcal{D}$,

$$\varinjlim \mathrm{sum}(\mathcal{D}) = \varinjlim \mathrm{add}(\mathcal{D}) \subseteq \varinjlim \mathrm{Add}(\mathcal{D}) = \varinjlim \mathrm{Sum}(\mathcal{D}) \subseteq \widetilde{\mathrm{Add}(\mathcal{D})}$$

where $\widetilde{\mathrm{Add}(\mathcal{D})}$ is the class of all pure-epimorphic images of the modules in $\mathrm{Add}(\mathcal{D})$.

Easy facts

- Since $\mathrm{Sum}(\mathcal{D}) \subseteq \varinjlim \mathrm{sum}(\mathcal{D})$, the equality in the first inclusion holds in case $\varinjlim \mathrm{add}(\mathcal{D})$ is closed under direct limits.
- Equality in the second inclusion just says that $\mathcal{L} = \varinjlim \mathrm{Add}(\mathcal{D})$ is closed under pure-epimorphic images. In this case, $\mathcal{L}$ is a covering class.

Failure of equality no. 2

Example

Let K be a field, R be the K-algebra of all eventually constant sequences in the K-algebra $Q = K^\omega$ of all sequences of elements of K. Then Q is the maximal quotient ring of R, and

$$\mathrm{Mod}\text{–}Q = \varinjlim \mathsf{add}(Q) = \varinjlim \mathsf{Add}(Q) \subsetneq \widetilde{\mathsf{Add}(Q)} = \mathsf{Gen}(Q_R).$$

Here, $\mathrm{Mod}\text{–}Q$ is not a full subcategory of $\mathrm{Mod}\text{–}R$, and it is not closed under direct summands in $\mathrm{Mod}\text{–}R$.

Hence, $\mathrm{Mod}\text{–}Q$ is not closed under direct limits in $\mathrm{Mod}\text{–}R$.

Sufficient conditions for equality no. 1

Lemma

Assume that $\mathcal{D}$ consists of small modules (or $\mathcal{D} = \{M\}$ for a self-small module). Then $\varinjlim add(\mathcal{D}) = \varinjlim Add(\mathcal{D})$.

Further positive cases

The equality holds when

- $\mathcal{D}$ is a class of injective modules over a right noetherian ring R.
- $\mathcal{D} = \{T^{-1}R/R\}$ where R is left noetherian and T is a countable multiplicative set of (some) central elements of R.
- ...

The case of projective modules

Open problem

Let P be a projective module. Does $\varinjlim \operatorname{add}(P) = \varinjlim \operatorname{Add}(P)$?

By the Lemma above, the equality holds when P is a direct sum of finitely generated modules.

[Příhoda]

$\operatorname{Add}(P) \subseteq \varinjlim \operatorname{add}(P)$ for each projective module P.
That is, $\varinjlim \operatorname{add}(P)$ and $\varinjlim \operatorname{Add}(P)$ contain the same projective modules.

Countably generated projective modules

Example: Continuous real functions

Let $R = C_{\langle 0,1\rangle}$ be the ring of all continuous real functions on $\langle 0, 1\rangle$. Then each countably generated pure ideal in R is projective. Moreover, pure ideals P of R correspond 1-1 to closed subsets of $\langle 0, 1\rangle$ via the mutually inverse assignments

$$\varphi : P \mapsto X = \bigcap_{f \in P} f^{-1}(0)$$

$$\phi : X \mapsto \{f \in R \mid f^{-1}(0) \text{ contains some open neighborhood of } X\}.$$

Countably generated projective modules

Let P be a pure ideal in $R = C_{\langle 0,1 \rangle}$ which is not finitely generated.

- There is a countable set I such that $P = \bigoplus_{i \in I} P_i$ and
$$\varphi(P_i) = \langle 0,1 \rangle \setminus O_i,$$
where $\{O_i \mid i \in I\}$ is a set of pairwise disjoint open intervals in $\langle 0,1 \rangle$.
- P is not self-small.
- $S = \mathrm{End}(P) \cong \prod_{i \in I} S_i$, where $S_i = C_{O_i}$, the ring of all continuous real functions on O_i.
- $\varinjlim \mathrm{add}(P) = \varinjlim \mathrm{Add}(P) = \{F \in \mathrm{Mod}\text{–}R \mid F \in \mathcal{F}_S \text{ and } F.P = F\}$.

Kaplansky's example is the particular case of $X = \{0\}$. Here, $P = \psi(X)$ is an indecomposable countably generated projective module, and $S = C_{\langle 0,1 \rangle} \supsetneq R$.

Countably generated pure ideals in commutative rings

Theorem

Let R be a commutative ring and P a countably generated pure ideal in R ($=$ trace ideal of a countably generated projective module).
Let $S = \mathrm{End}(P)$. Then P is projective, and

$$\varinjlim \mathrm{add}(P) = \varinjlim \mathrm{Add}(P) = \{F \in \mathrm{Mod}\text{–}R \mid F \in \mathcal{F}_S\,,\ F.P = F\} = \widetilde{\mathrm{Add}(P)}$$

is a covering class.

The tilting case, and more ...

[Šaroch'18], [Angeleri-Šaroch-T.'18]

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Then

- $\mathcal{B}$ is a definable class.
- $\mathrm{Ker}(\mathfrak{C}) = \mathcal{A} \cap \mathcal{B} = \mathrm{Add}(K)$ for a module K.
- $\widetilde{\mathrm{Add}(K)} = \widetilde{\mathcal{A}} \cap \mathcal{B}$.
- $\widetilde{\mathcal{A}} = \varinjlim(\widetilde{\mathcal{A}}^{\leq\omega})$.

Tilting as a special case: $\mathfrak{C}$ is tilting (that is, $\mathcal{B} = T^{\perp_\infty}$ for a tilting module T), iff moreover $\mathfrak{C}$ is hereditary, and $\mathcal{A}$ consists of modules of bounded projective dimension.

If $\mathfrak{C}$ is tilting, then we can take $K = T$, and T determines $\mathfrak{C}$. In this case even $\widetilde{\mathcal{A}} = \varinjlim(\widetilde{\mathcal{A}}^{<\omega}) = \varinjlim \mathcal{A}^{<\omega}$.

A 1-tilting example

Let R be a regular local ring of Krull dimension 2. Let $\mathcal{S}$ be the set of all ideals of R. Let $\mathcal{A} = \mathsf{Filt}(\mathcal{S})$ and $\mathcal{B} = \mathcal{I}_1$.

Then $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair induced by a countably generated 1-tilting module T.

$\widetilde{\mathcal{A}} = \varinjlim \mathcal{A}^{<\omega}$. However, the definable class $\mathcal{B}$ contains no finitely generated modules, and the same is true of the class $\mathsf{Ker}(\mathfrak{C}) = \mathsf{Add}(T)$.

Countably presented modules

Let C be a countably presented module in $\widetilde{\mathrm{Add}(K)}$.

Then there exist a countably presented module $D \in \mathrm{Add}(K)$ such that $C \oplus D$ is a countable direct limit of modules from $\mathrm{Add}(K)$.

In particular, if $\varinjlim \mathrm{Add}(K)$ is closed under direct summands, then $C \in \varinjlim \mathrm{Add}(K)$.

$\varinjlim \mathbf{Add}(K)$ versus $\widetilde{\mathbf{Add}(K)}$

Assume that either R is countable, or K is a direct sum of countably generated modules.
Then $\widetilde{\mathrm{Add}(K)} = \varinjlim \widetilde{\mathrm{Add}(K)}^{\leq \omega}$.

Theorem

Assume that the class $\varinjlim \mathrm{Add}(K)$ is closed under direct limits and either R is countable, or K is a direct sum of countably generated modules.
Then $\varinjlim \mathrm{Add}(K) = \widetilde{\mathrm{Add}(K)}$ is a covering class.

Further references

- L. Angeleri Hügel, J. Šaroch, J. Trlifaj, *Approximations and Mittag-Leffler conditions - the applications*, Israel J. Math. 226(2018), 757-780.
- S. Bazzoni, L. Positselski, J. Šťovíček, *Projective covers of flat contramodules*, Int. Math. Res. Not. IMRN (2021), DOI:10.1093/imrn/rnab202.
- L. Positselski, J. Šťovíček, *The tilting-cotilting correspondence*, Int. Math. Res. Not. 2021, 189-274.
- J. Šaroch *Approximations and Mittag-Leffler conditions - the tools*, Israel J. Math. 226(2018), 737-756.

Further references

- G. De Marco, *Projectivity of pure ideals*, Rend. Sem. Mat. Univ. Padova 68(1983), 423-430.
- W. Wm. McGovern, G. Puninski, P. Rothmaler, *When every projective module is a direct sum of finitely generated modules*, J. Algebra 315(2007), 454-481.
- D. Herbera, P. Příhoda, *Reconstructing projective modules from its trace ideal*, J. Algebra 416(2014), 25-57.