APPROXIMATIONS OF MODULES

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It is a well-known fact that the category of all modules, Mod–R, over a general associative ring R is too complex to admit classification. Unless R is of finite representation type we have to limit attempts at classification to some restricted subcategories of modules.

In order to overcome this problem, the approximation theory of modules has been developed over the past several decades. The idea is to select suitable subcategories $\mathcal{C}$ of Mod–R whose modules can be classified, and then approximate arbitrary modules by the ones from $\mathcal{C}$. These approximations are neither unique nor functorial in general, but there is a rich supply available appropriate to the requirements of various particular applications. Approximation theory has thus become an indispensable part of the classification theory of modules.

In these notes, we use set-theoretic homological algebra to develop elements of the approximation theory of modules. We introduce the basic notion of a filtration, present a powerful tool for working with filtrations, the Hill Lemma, and some of its recent applications. We also explain how cotorsion pairs provide for a variety of approximations; in particular, we prove the Flat Cover Conjecture. Finally, we show that all deconstructible classes of modules are precovering.

1. Filtrations

We start by introducing the basic notion of a filtration of a module:

Definition 1.1. Let R be a ring, M a module, and $\mathcal{C}$ a class of modules.

A chain of submodules, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of M is called *continuous*, provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \sigma$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

A continuous chain $\mathcal{M}$ is a *$\mathcal{C}$-filtration* of M, provided that $M = M_\sigma$, and each of the modules $M_{\alpha+1}/M_\alpha$ ($\alpha < \sigma$) is isomorphic to some element of $\mathcal{C}$.

If M is a $\mathcal{C}$-filtered module, then M is also called a *transfinite extension* of the modules in $\mathcal{C}$. A class $\mathcal{A}$ is said to be *closed under transfinite extensions* provided that $\mathcal{A}$ contains all $\mathcal{A}$-filtered modules. Clearly, this implies that $\mathcal{A}$ is closed under extensions and arbitrary direct sums.

M is called *$\mathcal{C}$-filtered*, provided that M possesses at least one $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. If σ can be taken finite, then M is called *finitely $\mathcal{C}$-filtered*.

We will use the notation $\mathrm{Filt}(\mathcal{C})$ for the class of all $\mathcal{C}$-filtered modules.

For example, if $\mathcal{C} = \operatorname{simp} R$ is a representative set of all simple modules, then the $\mathcal{C}$-filtered modules coincide with the semiartinian modules, while the finitely $\mathcal{C}$-filtered modules are exactly the modules of finite length. The latter modules are the subject of classic Jordan-Hölder theory. In a sense, the Hill Lemma and its applications presented below are extensions of this theory to the infinite setting where no dimensions and direct sum decompositions are available in general.

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For a class of modules $\mathcal{C}$, we denote by ${}^\perp\mathcal{C}$ the common kernel of all the contravariant Ext functors induced by the elements of $\mathcal{C}$, that is,

$${}^\perp\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{C}) = \{A \in \text{Mod–}R \mid \operatorname{Ext}^1_R(A,C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

The classes of the form ${}^\perp\mathcal{C}$ provide for a rich supply of the classes closed under transfinite extensions, as shown by Eklof [6]:

Lemma 1.2. (Eklof Lemma) *Let $\mathcal{C}$ be a class of modules. Then the class ${}^\perp\mathcal{C}$ is closed under transfinite extensions. That is, if M is a ${}^\perp\mathcal{C}$-filtered module, then $M \in {}^\perp\mathcal{C}$.*

Proof. It suffices to prove the claim for the case when $\mathcal{C} = \{N\}$ for a single module N.

Let $(M_\alpha \mid \alpha \leq \kappa)$ be a ${}^\perp N$-filtration of M. So $\operatorname{Ext}^1_R(M_0,N) = 0$ and, for each $\alpha < \kappa$, $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$. We will prove $\operatorname{Ext}^1_R(M,N) = 0$.

By induction on $\alpha \leq \kappa$ we will prove that $\operatorname{Ext}^1_R(M_\alpha,N) = 0$. This is clear for $\alpha = 0$.

The exact sequence

$$0 = \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \to \operatorname{Ext}^1_R(M_{\alpha+1},N) \to \operatorname{Ext}^1_R(M_\alpha,N) = 0$$

proves the induction step.

Assume $\alpha \leq \kappa$ is a limit ordinal. Let $0 \to N \to I \xrightarrow{\pi} I/N \to 0$ be an exact sequence with I an injective module. In order to prove that $\operatorname{Ext}^1_R(M_\alpha,N) = 0$, we show that the abelian group homomorphism $\operatorname{Hom}_R(M_\alpha,\pi) : \operatorname{Hom}_R(M_\alpha,I) \to \operatorname{Hom}_R(M_\alpha,I/N)$ is surjective.

Let $\varphi \in \operatorname{Hom}_R(M_\alpha,I/N)$. We now define by induction homomorphisms $\psi_\beta \in \operatorname{Hom}_R(M_\beta,I)$, $\beta < \alpha$, so that $\varphi \restriction M_\beta = \pi\psi_\beta$ and $\psi_\beta \restriction M_\gamma = \psi_\gamma$ for all $\gamma < \beta < \alpha$.

First define $M_{-1} = 0$ and $\psi_{-1} = 0$. If ψ_β is already defined, the injectivity of I yields the existence of $\eta \in \operatorname{Hom}_R(M_{\beta+1},I)$, such that $\eta \restriction M_\beta = \psi_\beta$. Put $\delta = \varphi \restriction M_{\beta+1} - \pi\eta \in \operatorname{Hom}_R(M_{\beta+1},I/N)$. Then $\delta \restriction M_\beta = 0$. Since $\operatorname{Ext}^1_R(M_{\beta+1}/M_\beta,N) = 0$, there is $\epsilon \in \operatorname{Hom}_R(M_{\beta+1},I)$, such that $\epsilon \restriction M_\beta = 0$ and $\pi\epsilon = \delta$. Put $\psi_{\beta+1} = \eta+\epsilon$. Then $\psi_{\beta+1} \restriction M_\beta = \psi_\beta$ and $\pi\psi_{\beta+1} = \pi\eta+\delta = \varphi \restriction M_{\beta+1}$. For a limit ordinal $\beta < \alpha$, put $\psi_\beta = \bigcup_{\gamma<\beta}\psi_\gamma$.

Finally, put $\psi_\alpha = \bigcup_{\beta<\alpha}\psi_\beta$. By the construction, $\pi\psi_\alpha = \varphi$.
The claim is just the case of $\alpha = \kappa$. □

2. Tools for dealing with filtrations

When studying a particular $\mathcal{C}$-filtered module, we often need to replace the original $\mathcal{C}$-filtration by another one that better fits the study in case. A remarkable construction serving this purpose was discovered by Hill [12]. It expands a given $\mathcal{C}$-filtration, $\mathcal{M}$, of a module M into a large family, $\mathcal{F}$, consisting of $\mathcal{C}$-filtered submodules of M. Moreover, $\mathcal{F}$ inherits the key property of $\mathcal{M}$: it forms a complete distributive sublattice of the modular lattice of all submodules of M.

The key notion here is that of a closed subset of the length of a $\mathcal{C}$-filtration:

Definition 2.1. Let R be a ring and $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a continuous chain of modules. Consider a family of modules $(A_\alpha \mid \alpha < \sigma)$, such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$.
A subset S of the ordinal σ is *closed*, if every $\alpha \in S$ satisfies

$$M_\alpha \cap A_\alpha \subseteq \sum_{\beta\in S,\beta<\alpha} A_\beta.$$

The *height* $\mathrm{hgt}(x)$ of an element $x \in M_\sigma$ is defined as the least ordinal $\alpha < \sigma$, such that $x \in M_{\alpha+1}$. For any subset S of σ, we define

$$M(S) = \sum_{\alpha \in S} A_\alpha.$$

For each ordinal $\alpha \leq \sigma$, we have $M_\alpha = \sum_{\beta<\alpha} A_\beta$, so α $(= \{\beta < \sigma \mid \beta < \alpha\})$ is a closed subset of σ.

Lemma 2.2. *Let S be a closed subset of σ and $x \in M(S)$. Let $S' = \{\alpha \in S \mid \alpha \leq hgt(x)\}$. Then $x \in M(S')$.*

Proof. Let $x \in M(S)$. Then $x = x_1 + \cdots + x_k$, where $x_i \in A_{\alpha_i}$ for some $\alpha_i \in S$, $1 \leq i \leq k$. W.l.o.g., $\alpha_1 < \cdots < \alpha_k$, and α_k is minimal.

If $\alpha_k > \mathrm{hgt}(x)$, then $x_k = x - x_1 - \cdots - x_{k-1} \in M_{\alpha_k} \cap A_{\alpha_k} \subseteq \sum_{\alpha \in S, \alpha < \alpha_k} A_\alpha$, since S is closed, in contradiction with the minimality of α_k. □

As an immediate corollary, we have

Corollary 2.3. *Let S be a closed subset of σ and $x \in M(S)$. Then $hgt(x) \in S$.*

Lemma 2.4. *Let $(S_i \mid i \in I)$ be a family of closed subsets of σ. Then*

$$M\big(\bigcap_{i\in I} S_i\big) = \bigcap_{i \in I} M(S_i) \quad \text{ and } \quad M\big(\bigcup_{i\in I} S_i\big) = \sum_{i\in I} M(S_i).$$

Proof. For the first equality, let $T = \bigcap_{i\in I} S_i$. Clearly, $M(T) \subseteq \bigcap_{i\in I} M(S_i)$. Suppose there is an $x \in \bigcap_{i \in I} M(S_i)$, such that $x \notin M(T)$, and choose such an x of minimal height. Then $x = y + z$ for some $y \in A_{\mathrm{hgt}(x)}$ and $z \in M_{\mathrm{hgt}(x)}$. By Corollary 2.3, $\mathrm{hgt}(x) \in S_i$ for all $i \in I$, so $\mathrm{hgt}(x) \in T$ and $y \in M(T)$. Then $z \in \bigcap_{i\in I} M(S_i)$, $z \notin M(T)$ and $\mathrm{hgt}(z) < \mathrm{hgt}(x)$, in contradiction to minimality.

The second equality is immediate from Definition 2.1. □

Remark 2.5. Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a continuous chain of modules. If N is a submodule of $M = M_\sigma$, then $\mathcal{M}$ induces the continuous chain of submodules $\mathcal{N} = (N \cap M_\alpha \mid \alpha \leq \sigma)$ of N.

If, moreover, $N = M(S)$ for a subset $S \subseteq \sigma$, then another continuous chain of submodules of N is given by $\mathcal{N}' = (M(S \cap \alpha) \mid \alpha \leq \sigma)$.

Notice that the set S is closed in σ, if and only if the chains $\mathcal{N}$ and $\mathcal{N}'$ coincide. The only-if part holds, because $M(S) \cap M_\alpha = M(S) \cap M(\alpha) = M(S \cap \alpha)$ for each $\alpha \leq \sigma$ by Lemma 2.4. Conversely, if $\alpha \in S$, then $M_\alpha \cap A_\alpha \subseteq M(\alpha) \cap M(S) = M(S \cap \alpha) = \sum_{\beta \in S, \beta < \alpha} A_\beta$.

Next we prove that intersections and unions of closed subsets are again closed:

Proposition 2.6. *Let $(S_i \mid i \in I)$ be a family of closed subsets of σ. Then both the union and the intersection of this family are again closed in σ.*

Proof. As for the union, if $\beta \in S = \bigcup_{i\in I} S_i$, then $\beta \in S_i$ for some $i \in I$ and $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_i, \alpha < \beta} A_\alpha \subseteq \sum_{\alpha \in S, \alpha<\beta} A_\alpha$.

For the intersection, let $\beta \in T = \bigcap_{i \in I} S_i$. Then $M_\beta \cap A_\beta \subseteq M(S_i \cap \beta)$ for each $i \in I$. Therefore Lemma 2.4 implies that

$$M_\beta \cap A_\beta \subseteq \bigcap_{i\in I} M(S_i \cap \beta) = M(T \cap \beta),$$

which exactly says that T is closed. □

By Proposition 2.6, closed subsets form a complete sublattice $C(\sigma)$ of the complete Boolean lattice of all subsets of σ.

Assume that the chain $\mathcal{M}$ is strictly increasing, and $S, S' \in C(\sigma)$. Then $S \subseteq S'$, if and only if $M(S) \subseteq M(S')$. The only-if-part is trivial; to prove the if-part, assume that $M(S) \subseteq M(S')$ and there is a least ordinal α in $S \setminus S'$. Then $A_\alpha \subseteq M(S \cap (\alpha + 1)) = M(S) \cap M(\alpha+1) \subseteq M(S') \cap M(\alpha+1) = M(S' \cap (\alpha+1)) = M(S' \cap \alpha) \subseteq M_\alpha$, a contradiction.

Let $M = M_\sigma$, and let $L(M)$ denote the lattice of all submodules of M. We can summarize the above as

Corollary 2.7. *Assume that the chain $\mathcal{M}$ is strictly increasing. Then the map $\theta : S \mapsto M(S)$ is a complete lattice isomorphism of the complete distributive lattice $C(\sigma)$ onto a sublattice of the complete modular lattice $L(M)$.*

Even if $\mathcal{M}$ is not strictly increasing, being a homomorphic image of a distributive lattice, the image of θ is still a distributive sublattice of $L(M)$. This image yields the desired family of submodules $\mathcal{F}$, extending the given continuous chain $\mathcal{M}$. Basic properties of $\mathcal{F}$ are the subject matter of the general version of the Hill Lemma from [18]:

Theorem 2.8. (Hill Lemma) *Let R be a ring, κ an infinite regular cardinal, and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let M be a module with a $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$. Then there is a family $\mathcal{F}$ consisting of submodules of M such that*

- (H1) *$\mathcal{M} \subseteq \mathcal{F}$.*
- (H2) *$\mathcal{F}$ is closed under arbitrary sums and intersections. $\mathcal{F}$ is a complete distributive sublattice of the modular lattice of all submodules of M.*
- (H3) *Let $N, P \in \mathcal{F}$ be, such that $N \subseteq P$. Then the module P/N is $\mathcal{C}$-filtered.*

  Moreover, there exist an ordinal $\tau \leq \sigma$ and a continuous chain $(F_\gamma \mid \gamma \leq \tau)$ of elements of $\mathcal{F}$, such that $\mathcal{Q} = (F_\gamma/N \mid \gamma \leq \tau)$ is a $\mathcal{C}$-filtration of P/N, and for each $\gamma < \tau$ there is a $\beta < \sigma$ with $F_{\gamma+1}/F_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.
- (H4) *Let $N \in \mathcal{F}$ and X be a subset of M of cardinality $< \kappa$. Then there is a $P \in \mathcal{F}$, such that $N \cup X \subseteq P$ and P/N is $< \kappa$-presented.*

Proof. Consider a family of $< \kappa$-generated modules $(A_\alpha \mid \alpha < \sigma)$, such that for each $\alpha < \sigma$:

$$M_{\alpha+1} = M_\alpha + A_\alpha,$$

as in Definition 2.1. We claim that

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$$

has properties (H1)-(H4).

Property (H1) is clear, since each ordinal $\alpha \leq \sigma$ is a closed subset of σ.

The first claim in (H2) follows by Proposition 2.6 and Lemma 2.4, the second by Corollary 2.7, because $\mathcal{F}$ is the image of θ.

Property (H3) is proved as follows: we have $N = M(S)$ and $P = M(T)$ for some closed subsets S, T. Since $S \cup T$ is closed, we can w.l.o.g. assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha = M(S \cup (T \cap \beta)) \qquad \text{and} \qquad \bar{F}_\beta = F_\beta/N.$$

Clearly $(\bar{F}_\beta \mid \beta \leq \sigma)$ is a filtration of P/N, such that $\bar{F}_{\beta+1} = \bar{F}_\beta + (A_\beta + N)/N$ for $\beta \in T \setminus S$ and $\bar{F}_{\beta+1} = \bar{F}_\beta$ otherwise. Let $\beta \in T \setminus S$. Then

$$\bar{F}_{\beta+1}/\bar{F}_\beta \cong F_{\beta+1}/F_\beta \cong A_\beta/(F_\beta \cap A_\beta),$$

and

$$F_\beta \cap A_\beta \supseteq \Big(\sum_{\alpha \in T, \alpha < \beta} A_\alpha \Big) \cap A_\beta = M_\beta \cap A_\beta.$$

However, if $x \in F_\beta \cap A_\beta$, then $\mathrm{hgt}(x) \leq \beta$, so $x \in M(T')$ by Lemma 2.2, where $T' = \{\alpha \in S \cup (T \cap \beta) \mid \alpha \leq \beta\}$. By Proposition 2.6, we get $x \in M_\beta$, because $\beta \notin S$. Hence $F_\beta \cap A_\beta = M_\beta \cap A_\beta$ and $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$. The desired $\mathcal{C}$-filtration $\mathcal{Q}$ of P/N is obtained from $(\bar{F}_\beta \mid \beta \leq \sigma)$ by removing possible repetitions, and (H3) follows. Denote by τ' the ordinal type of the well-ordered set $(T \setminus S, <)$. Notice that the length τ of the filtration can be taken as $1 + \tau'$ (the ordinal sum, hence $\tau = \tau'$ for τ' infinite).

For property (H4) we first prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. Because κ is an infinite regular cardinal, by Proposition 2.6, it is enough to prove this only for one-element subsets of σ. So we prove that every $\beta < \sigma$ is contained in a closed subset of cardinality $< \kappa$, by induction on β. For $\beta < \kappa$, we just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \to M_\beta \cap A_\beta \to A_\beta \to M_{\beta+1}/M_\beta \to 0$$

shows that $M_\beta \cap A_\beta$ is $< \kappa$-generated. Thus $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \kappa$. Moreover, we can assume that S_0 is closed in σ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that S is closed, it suffices to check the definition only for β. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$, where S is closed in σ, and let X be a subset of M of cardinality $< \kappa$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset T of σ of cardinality $< \kappa$. By the preceding paragraph, we can assume that T is closed in σ. Let $P = M(S \cup T)$. Then P/N is $\mathcal{C}$-filtered by property (H3), and the filtration can be chosen indexed by $1+$ the ordinal type of $T \setminus S$, which is certainly less than κ. In particular, P/N is $< \kappa$-presented. □

Remark 2.9. If we assume the stronger assumption that each module in $\mathcal{C}$ possesses a projective resolution consisting of $< \kappa$-generated modules, then the same is true of the module P/N in (H4).

3. Approximations

Throughout this section we assume that R is a ring, M is a (right R-) module and $\mathcal{C}$ a class of modules closed under isomorphic images and direct summands.

Definition 3.1. A map $f \in \mathrm{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-preenvelope* of M, provided that the map $\mathrm{Hom}_R(f, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. That is, for each homomorphism $f' : M \to C'$ there is a homomorphism $g : C \to C'$, such that $f' = gf$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f'"'] & C \arrow[d, dashed, "g"] \\ & C'. \end{tikzcd}$$

(Note that we require the existence, but not the uniqueness, of the map g.)
The $\mathcal{C}$-preenvelope f is a *$\mathcal{C}$-envelope* of M. provided that f is *left minimal*, that is, provided $f = gf$ implies g is an automorphism for each $g \in \mathrm{End}_R(C)$.

Example 3.2. The embedding $M \hookrightarrow E(M)$ is easily seen to be the $\mathcal{I}_0$-envelope of a module M.

Clearly a $\mathcal{C}$-envelope of M is unique in the following sense: if $f : M \to C$ and $f' : M \to C'$ are $\mathcal{C}$-envelopes of M, then there is an isomorphism $g : C \to C'$, such that $f' = gf$.

In general a module M may have many non-isomorphic $\mathcal{C}$-preenvelopes, but no $\mathcal{C}$-envelope. Nevertheless, if the $\mathcal{C}$-envelope exists, its minimality implies that it is isomorphic to a direct summand in each $\mathcal{C}$-preenvelope of M:

Lemma 3.3. *Let $f : M \to C$ be a $\mathcal{C}$-envelope and $f' : M \to C'$ a $\mathcal{C}$-preenvelope of a module M. Then*

(a) *$C' = D \oplus D'$, where $\operatorname{Im} f' \subseteq D$ and $f' : M \to D$ is a $\mathcal{C}$-envelope of M;*
(b) *f' is a $\mathcal{C}$-envelope of M, if and only if C' has no proper direct summands containing $\operatorname{Im} f'$.*

Proof. (a) By definition there are homomorphisms $g : C \to C'$ and $g' : C' \to C$, such that $f' = gf$ and $g'g$ is an automorphism of C. So $D = \operatorname{Im} g \cong C$ is a direct summand in C' containing $\operatorname{Im} f'$, and the assertion follows.
(b) by part (a). $\square$

Definition 3.4. A class $\mathcal{C} \subseteq$ Mod–R is a *preenveloping class*, (*enveloping class*) provided that each module has a $\mathcal{C}$-preenvelope ($\mathcal{C}$-envelope).

For example, the class $\mathcal{I}_0$ of all injective modules from Example 3.2 is an enveloping class of modules.

Now we briefly discuss the dual concepts:

Definition 3.5. A map $f \in \operatorname{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of M, provided that the abelian group homomorphism $\operatorname{Hom}_R(C', f) : \operatorname{Hom}_R(C', C) \to \operatorname{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$.
A $\mathcal{C}$-precover $f \in \operatorname{Hom}_R(C, M)$ of M is called a *$\mathcal{C}$-cover* of M, provided that f is *right minimal*, that is, provided $fg = f$ implies that g is an automorphism for each $g \in \operatorname{End}_R(C)$.
$\mathcal{C} \subseteq$ Mod–R is a *precovering class*, (*covering class*) provided that each module has a $\mathcal{C}$-precover ($\mathcal{C}$-cover).

Remark 3.6. $\mathcal{C}$-preenvelopes and $\mathcal{C}$-precovers are also called *left* and *right approximations.*

If Mod–R is replaced by its subcategory mod–R in the definitions above, then preenveloping and precovering classes are called *covariantly finite* and *contravariantly finite*, respectively (cf. [3] and [4]).

Example 3.7. Each module M has a $\mathcal{P}_0$-precover (where $\mathcal{P}_0$ denotes the class of all projective modules), since each module is a homomorphic image of a projective module. Moreover, M has a $\mathcal{P}_0$-cover, if and only if M has a projective cover in the sense of Bass (that is, there is an epimorphism $f : P \to M$ with P projective and $\operatorname{Ker}(f)$ a small submodule of P). So $\mathcal{P}_0$ is always a precovering class, and it is a covering class, if and only if R is a right perfect ring.

$\mathcal{C}$-covers may not exist in general, but if they exist, they are unique up to isomorphism. As in Lemma 3.3, we get

Lemma 3.8. *Let $f : C \to M$ be the $\mathcal{C}$-cover of M. Let $f' : C' \to M$ be any $\mathcal{C}$-precover of M. Then*

(a) *$C' = D \oplus D'$, where $D \subseteq \operatorname{Ker} f'$ and $f' \restriction D'$ is a $\mathcal{C}$-cover of M.*
(b) *f' is a $\mathcal{C}$-cover of M, if and only if C' has no non-zero direct summands contained in $\operatorname{Ker} f'$.*

Proof. Dual to the proof of Lemma 3.3. □

The following lemma is known as the Wakamatsu Lemma (see [19]). It shows that under rather weak assumptions on the class $\mathcal{C}$, $\mathcal{C}$-envelopes and $\mathcal{C}$-covers are special in the sense of the following definition:

Definition 3.9. Let $\mathcal{C} \subseteq \text{Mod–}R$. We define

$$\mathcal{C}^{\perp} = \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{C}, -) = \{N \in \text{Mod–}R \mid \operatorname{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\}.$$

For $\mathcal{C} = \{C\}$, we write for short $C^{\perp}$ and ${}^{\perp}C$ in place of $\{C\}^{\perp}$ and ${}^{\perp}\{C\}$, respectively.
Let $M \in \text{Mod–}R$. A $\mathcal{C}$-preenvelope $f : M \to C$ of M is called *special*, provided that f is injective and $\operatorname{Coker} f \in {}^{\perp}\mathcal{C}$. So a special $\mathcal{C}$-preenvelope may be viewed as an exact sequence

$$0 \to M \xrightarrow{f} C \to D \to 0$$

with $C \in \mathcal{C}$ and $D \in {}^{\perp}\mathcal{C}$.
Dually, a $\mathcal{C}$-precover $f : C \to M$ of M is called *special*, if f is surjective and $\operatorname{Ker} f \in \mathcal{C}^{\perp}$.
If $\mathcal{C}$ is a class of modules such that each module M has a special $\mathcal{C}$-preenvelope (special $\mathcal{C}$-precover), then $\mathcal{C}$ is called *special preenveloping* (*special precovering*).

Lemma 3.10. (Wakamatsu Lemma) *Let $M \in \text{Mod–}R$ and $\mathcal{C} \subseteq \text{Mod–}R$ be a class closed under extensions.*

(a) *Let $f : M \to C$ be a monic $\mathcal{C}$-envelope of M. Then f is special.*
(b) *Let $f : C \to M$ be a surjective $\mathcal{C}$-cover of M. Then f is special.*

Proof. (a) By assumption, there is an exact sequence

$$0 \to M \xrightarrow{f} C \xrightarrow{g} D \to 0.$$

In order to prove that $D \in {}^{\perp}\mathcal{C}$, we take an arbitrary extension

$$0 \to C' \to X \xrightarrow{h} D \to 0$$

with $C' \in \mathcal{C}$. We will prove that h splits. First consider the pullback of g and h:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & P \arrow[r, "\beta"] \arrow[d, "\gamma"'] & X \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & M \arrow[r, "f"] & C \arrow[r, "g"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $C, C' \in \mathcal{C}$, also $P \in \mathcal{C}$ by assumption. Since f is a $\mathcal{C}$-envelope of M, there is a homomorphism $\delta : C \to P$ with $\alpha = \delta f$. Then $f = \gamma\alpha = \gamma\delta f$, so $\gamma\delta$ is an automorphism of C.

Define $i : D \to X$ by $i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$. This is well-defined, since

$$\delta(\gamma\delta)^{-1} f(m) = \delta f(m) = \alpha(m).$$

Moreover, $hig = h\beta\delta(\gamma\delta)^{-1} = g\gamma\delta(\gamma\delta)^{-1} = g$, so $hi = \mathrm{id}_D$ and h splits.

(b) is dual to (a). $\square$

Remark 3.11. The $\mathcal{C}$-envelope f of a module M must be monic, provided that $\mathcal{I}_0 \subseteq \mathcal{C}$. This is because $M \hookrightarrow E(M)$ factors through f. Similarly, $\mathcal{P}_0 \subseteq \mathcal{C}$ implies that any $\mathcal{C}$-cover of M is surjective.

Also notice that the Wakamatsu Lemma holds with Mod–R replaced by its subcategory of all finitely presented modules mod–R.

4. Cotorsion pairs

Besides the Wakamatsu Lemma, there is another reason for investigating special preenvelopes and precovers, namely the existence of an explicit duality between them arising from the notion of a cotorsion pair:

Definition 4.1. Let $\mathcal{A}, \mathcal{B} \subseteq$ Mod–R. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* if $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$.

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C} \subseteq {}^\perp(\mathcal{C}^\perp)$ as well as $\mathcal{C} \subseteq ({}^\perp\mathcal{C})^\perp$. Moreover, $\mathfrak{G}_\mathcal{C} = ({}^\perp(\mathcal{C}^\perp), \mathcal{C}^\perp)$ and $\mathfrak{C}_\mathcal{C} = ({}^\perp\mathcal{C}, ({}^\perp\mathcal{C})^\perp)$ are easily seen to be cotorsion pairs, called the cotorsion pairs *generated* and *cogenerated*, respectively, by the class $\mathcal{C}$.

For any ring R, the cotorsion pairs of right R-modules are partially ordered by inclusion of their first components. In fact, they form a complete lattice L_{Ext}.

The largest element of L_{Ext} is $\mathfrak{G}_{\mathrm{Mod}-R} = (\mathrm{Mod}\text{–}R, \mathcal{I}_0)$, while the least is $\mathfrak{C}_{\mathrm{Mod}-R} = (\mathcal{P}_0, \mathrm{Mod}\text{–}R)$. These are the *trivial cotorsion pairs.*

Note that $(\bigcap_{\alpha<\kappa} \mathcal{A}_\alpha, (\bigcap_{\alpha<\kappa} \mathcal{A}_\alpha)^\perp)$ is the infimum of a sequence of cotorsion pairs $\{(\mathcal{A}_\alpha, \mathcal{B}_\alpha) \mid \alpha < \kappa\}$ in L_{Ext}, while $({}^\perp((\bigcup_{\alpha<\kappa} \mathcal{A}_\alpha)^\perp), \bigcap_{\alpha<\kappa} \mathcal{B}_\alpha)$ is its supremum.

Cotorsion pairs are analogues of the classical (non-hereditary) torsion pairs, where Hom ($= \mathrm{Ext}^0$) is replaced by Ext^1. Similarly, one can define F-pairs for any additive bifunctor F on Mod–R.

Now we present several important examples of cotorsion pairs:

Example 4.2. For any ring R and any $n \geq 0$, there are cotorsion pairs $(\mathcal{P}_n, \mathcal{P}_n^\perp)$, $(\mathcal{F}_n, \mathcal{F}_n^\perp)$, and $({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ where $\mathcal{P}_n$, $\mathcal{F}_n$, and $\mathcal{I}_n$ denotes the class of all modules of projective (flat, injective) dimension $\leq n$, respectively.

If R is an integral domain, then there is a cotorsion pair $(\mathcal{TF}, \mathcal{TF}^\perp)$ where $\mathcal{TF}$ is the class of all torsionfree modules.

We now record an immediate corollary of Lemma 3.10:

Corollary 4.3. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. If $\mathcal{A}$ is covering, then $\mathcal{A}$ is special precovering, and if $\mathcal{B}$ is enveloping, then $\mathcal{B}$ is special preenveloping.*

The key property of cotorsion pairs is their relation to module approximations. This fact – discovered by Salce [16] – says that the mutually dual categorical notions of a special precover and a special preenvelope are tied up by the homological tie of a cotorsion pair. In a sense, this fact is a remedy for the non-existence of a duality involving the category of all modules over a ring.

Lemma 4.4. (Salce Lemma) *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules. Then the following are equivalent:*

(a) *Each module has a special $\mathcal{A}$-precover.*
(b) *Each module has a special $\mathcal{B}$-preenvelope.*

In this case, the cotorsion pair $\mathfrak{C}$ is called complete.

Proof. (a) implies (b): let $M \in \text{Mod–}R$. There is an exact sequence

$$0 \to M \to I \xrightarrow{\pi} F \to 0,$$

where I is injective. By assumption, there is a special $\mathcal{A}$-precover ρ of F

$$0 \to B \to A \xrightarrow{\rho} F \to 0.$$

Consider the pullback of π and ρ:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d] & M \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\gamma"'] & I \arrow[r] \arrow[d, "\pi"'] & 0 \\
0 \arrow[r] & B \arrow[r] & A \arrow[r, "\rho"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $B, I \in \mathcal{B}$, also $P \in \mathcal{B}$. So the left-hand vertical exact sequence is a special $\mathcal{B}$-preenvelope of M.

(b) implies (a): by a dual argument. □

5. The abundance of complete cotorsion pairs

The following theorem, showing that complete cotorsion pairs are abundant, was originally proved in [7]. Similar arguments have been used in homotopy theory since Quillen's fundamental work [15] under the name of *small object argument.* The proof presented here is a more categorical modification of the one in [7], coming from [1]:

Theorem 5.1. (Completeness of cotorsion pairs generated by sets) *Let $\mathcal{S}$ be a set of modules.*

(a) *Let M be a module. Then there is a short exact sequence*

$$0 \to M \hookrightarrow P \to N \to 0,$$

where $P \in \mathcal{S}^\perp$ and N is $\mathcal{S}$-filtered.

In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^\perp$-preenvelope of M.

(b) *The cotorsion pair $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is complete.*

Proof. (a) Put $X = \bigoplus_{S \in \mathcal{S}} S$. Then $X^\perp = \mathcal{S}^\perp$. So w.l.o.g., we assume that $\mathcal{S}$ consists of a single module S.

Let $0 \to K \xrightarrow{\mu} F \to S \to 0$ be a short exact sequence with F a free module. Let λ be an infinite regular cardinal, such that K is $< \lambda$-generated.

By induction we define an increasing chain $(P_\alpha \mid \alpha < \lambda)$ as follows:
First let $P_0 = M$. For $\alpha < \lambda$, choose the index set $I_\alpha = \operatorname{Hom}_R(K, P_\alpha)$. We define μ_α as the direct sum of $|I_\alpha|$ copies of the homomorphism μ, i.e.

$$\mu_\alpha := \mu^{(I_\alpha)} \in \operatorname{Hom}_R(K^{(I_\alpha)}, F^{(I_\alpha)}).$$

Then μ_α is a monomorphism, and $\operatorname{Coker} \mu_\alpha$ is isomorphic to a direct sum of copies of S. Let $\varphi_\alpha \in \operatorname{Hom}_R(K^{(I_\alpha)}, P_\alpha)$ be the canonical morphism. Note that for each $\eta \in I_\alpha$ there are canonical embeddings $\nu_\eta \in \mathrm{Hom}_R(K, K^{(I_\alpha)})$ and $\nu'_\eta \in \mathrm{Hom}_R(F, F^{(I_\alpha)})$, such that $\eta = \varphi_\alpha \nu_\eta$ and $\nu'_\eta \mu = \mu_\alpha \nu_\eta$.

Now $P_{\alpha+1}$ is defined via the pushout of μ_α and φ_α:

$$\begin{array}{ccc} K^{(I_\alpha)} & \xrightarrow{\mu_\alpha} & F^{(I_\alpha)} \\ {\scriptstyle\varphi_\alpha}\downarrow & & {\scriptstyle\psi_\alpha}\downarrow \\ P_\alpha & \xrightarrow{\subseteq} & P_{\alpha+1}. \end{array}$$

If $\alpha \leq \lambda$ is a limit ordinal, we put $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$, so the chain is continuous. Put $P = \bigcup_{\alpha<\lambda} P_\alpha$.

We will prove that $\nu : M \hookrightarrow P$ is a special $S^\perp$-preenvelope of M.
First we prove that $P \in S^\perp$. Since F is projective, we are left to show that any $\varphi \in \mathrm{Hom}_R(K, P)$ factors through μ:

Since K is $<\lambda$-generated, there are an index $\alpha < \lambda$ and $\eta \in I_\alpha$, such that $\varphi(k) = \eta(k)$ for all $k \in K$. The pushout square gives $\psi_\alpha \mu_\alpha = \sigma_\alpha \varphi_\alpha$, where σ_α denotes the inclusion of P_α into $P_{\alpha+1}$. Altogether we have $\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = \sigma_\alpha \varphi_\alpha \nu_\eta = \sigma_\alpha \eta$. It follows that $\varphi = \psi' \mu$, where $\psi' \in \mathrm{Hom}_R(F, P)$ is defined by $\psi'(f) = \psi_\alpha \nu'_\eta(f)$ for all $f \in F$. This proves that $P \in S^\perp$.

It remains to prove that $N = P/M \in {}^\perp(\mathcal{S}^\perp)$. By construction, N is the union of the continuous chain $(N_\alpha \mid \alpha < \lambda)$, where $N_\alpha = P_\alpha/M$.

Since $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of S by the pushout construction, so is $N_{\alpha+1}/N_\alpha \cong P_{\alpha+1}/P_\alpha$. Since $S \in {}^\perp(\mathcal{S}^\perp)$, Lemma 1.2 shows that $N \in {}^\perp(\mathcal{S}^\perp)$.

(b) follows by part (a) (cf. Lemma 4.4). □

Any cotorsion pair generated by a set of modules $\mathcal{S}$ is also generated by the single module $M = \bigoplus_{S\in\mathcal{S}} S$. So the following corollary of Theorem 5.1 provides a characterization of the (complete) cotorsion pairs generated by sets of modules:

Corollary 5.2. *Let M be a module. Denote by $\mathcal{C}_M$ the class of all modules C, such that there is an exact sequence $0 \to F \to C \to G \to 0$, where F is free and G is $\{M\}$-filtered. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The following are equivalent*

(a) *$\mathfrak{C}$ is generated by M (that is, $\mathcal{B} = M^\perp$).*
(b) *$\mathcal{A}$ consists of all direct summands of elements of $\mathcal{C}_M$ (and for each $A \in \mathcal{A}$, there are $C \in \mathcal{C}_M$ and $B \in \mathcal{K}_{\mathfrak{C}}$, such that $A \oplus B \cong C$).*

Proof. (a) implies (b): by assumption, $\mathcal{B} = M^\perp$. Take $A \in \mathcal{A}$, and let $0 \to N \xrightarrow{\mu} F \to A \to 0$ be a short exact sequence with F free. By Theorem 5.1 (a), there is a special $\mathcal{B}$-preenvelope, $\nu : N \hookrightarrow B$ of N, such that $G = B/N$ is $\{M\}$-filtered. Let $(G_\alpha \mid \alpha \leq \lambda)$ be an $\{M\}$-filtration of G. Consider the pushout of μ and ν:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\nu"'] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\pi"'] & A \arrow[r] & 0 \\
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

The second column gives $C \in \mathcal{C}_M$. The second row splits since $B \in \mathcal{B}$ and $A \in \mathcal{A}$, so $A \oplus B \cong C$. Finally, since $F, G \in \mathcal{A}$, we have $C \in \mathcal{A}$, so $B \in \mathcal{K}_{\mathfrak{C}}$.

(b) implies (a): by Lemma 1.2, $M^{\perp} = \mathcal{A}^{\perp} = \mathcal{B}$. □

Corollary 5.3. *Let $\mathcal{S}$ be a set of modules containing R. Then the class ${}^{\perp}(\mathcal{S}^{\perp})$ consists of all direct summands of $\mathcal{S}$-filtered modules.*

Proof. By Corollary 5.2 and Lemma 1.2. □

In general, we cannot omit the term "direct summands" in Corollary 5.3. For example, if $\mathcal{S} = \{R\}$, then ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{P}_0$ is the class of all projective modules, while $\mathcal{S}$-filtered modules coincide with the free modules. There is, however, a way of getting rid of the direct summands, as we will see in the following section.

6. Kaplansky theorem for cotorsion pairs

By Corollary 5.3, if $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair generated by a set $\mathcal{C}$ containing R, then $\mathcal{A}$ coincides with the class of all direct summands of $\mathcal{C}$-filtered modules. Our next goal is to remove the term "direct summands" from this characterization of $\mathcal{A}$ on the account of replacing the set $\mathcal{C}$ by a suitable small subset of $\mathcal{A}$.
We will make use of the following application of Theorem 2.8:

Lemma 6.1. *Let κ be an uncountable regular cardinal and $\mathcal{C}$ be a class of $< \kappa$-presented modules. Denote by $\mathcal{A}$ the class of all direct summands of $\mathcal{C}$-filtered modules, and by $\mathcal{A}^{<\kappa}$ the subclass of all $< \kappa$-presented modules from $\mathcal{A}$.*

Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.

Proof. Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$-filtered module M, such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{F}$ be the family of submodules of M, as in Theorem 2.8. We proceed in two steps:

Step I: By induction, we construct a continuous chain, $(N_\alpha \mid \alpha \leq \tau)$, of submodules of M, such that $N_\tau = M$ and

(a) $N_\alpha \in \mathcal{F}$,
(b) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and
(c) $N_{\alpha+1}/N_\alpha \in (\text{Mod-}R)^{<\kappa}$,

for each $\alpha < \tau$.

First $N_0 = 0$, and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneqq M$, and we wish to construct $N_{\alpha+1}$. Take $x \in M \setminus N_\alpha$; by property (H4), there is $Q_0 \in \mathcal{F}$, such that $N_\alpha \cup \{x\} \subseteq Q_0$ and $Q_0/N_\alpha \in (\text{Mod-}R)^{<\kappa}$. Let X_0 be a subset of Q_0 of cardinality $< \kappa$, such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates Q_0/N_α. Put $Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module Z_0/N_α is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus we can find $Q_1 \in \mathcal{F}$, such that $Z_0 \subseteq Q_1$ and $Q_1/N_\alpha \in (\text{Mod-}R)^{<\kappa}$. Similarly, we infer that Z_1/N_α is $< \kappa$-generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$ and find $Q_2 \in \mathcal{F}$ with $Z_1 \subseteq Q_2$ and $Q_2/N_\alpha \in (\text{Mod-}R)^{<\kappa}$. In this way we obtain a chain $Q_0 \subseteq Q_1 \subseteq \dots$, such that for all $i < \omega$: $Q_i \in \mathcal{F}$, $Q_i/N_\alpha \in (\text{Mod-}R)^{<\kappa}$ and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \bigcup_{i<\omega} Q_i$ satisfies the properties (a)-(c).

Step II: by condition (b), we have

$$\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$$

and similarly for L. Hence

$$\begin{aligned}
&(\pi_K(N_{\alpha+1}) + N_\alpha) \cap (\pi_L(N_{\alpha+1}) + N_\alpha) \\
&\quad = (\pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha)) \\
&\quad = \big(\pi_K(N_{\alpha+1}) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\big) \oplus \pi_L(N_\alpha) \\
&\quad = \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha
\end{aligned}$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (a), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$-filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$-filtered module, we infer that $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. Since the class of all $< \kappa$-presented modules is closed under direct summands, (c) yields that $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$-presented. So $(\pi_K(N_\alpha) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}^{<\kappa}$-filtration of $K = \pi_K(N_\tau)$. □

We can now state the main result of this section:

Theorem 6.2. (Kaplansky theorem for cotorsion pairs) *Let R be a ring, κ an uncountable regular cardinal and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair of R-modules. Then the following conditions are equivalent:*

(a) *$\mathfrak{C}$ is generated by a class $\mathcal{C}$, such that $\mathcal{C}$ consists of $< \kappa$-presented modules.*
(b) *Every module in $\mathcal{A}$ is $\mathcal{D}$-filtered, where $\mathcal{D}$ is the class of all $< \kappa$-presented modules in $\mathcal{A}$.*

Proof. (a) $\implies$ (b). W.l.o.g., $\mathcal{C}$ is a set and $R \in \mathcal{C}$. By Corollary 5.3, $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules. So statement (b) follows by Lemma 6.1.

(b) $\implies$ (a). By the Eklof Lemma 1.2, every $\mathcal{A}$-filtered module is again in $\mathcal{A}$. Thus (b) implies that $\mathfrak{C}$ is generated by the class $\mathcal{D}$. □

The name of Theorem 6.2 above comes from the fact that its application to the cotorsion pair $(\mathcal{P}_0, \text{Mod-}R)$ generated by R yields (for $\kappa = \aleph_1$) the following classic theorem on the structure of projective modules due to Kaplansky [13]:

Corollary 6.3. *Every projective module over an arbitrary ring is a direct sum of countably generated projective modules.*

Remark 6.4. The latter application also shows that in general it is not possible to extend Theorem 6.2 to the case of $\kappa = \aleph_0$. Namely, there exist rings R which admit countably generated projective modules that are not direct sums of finitely generated projective ones. (For example, following Kaplansky, consider the commutative ring R of all continuous real-valued functions on $[0,1]$ and its ideal P consisting of all functions $f \in R$ vanishing on some interval $[0, \epsilon(f)]$, where $\epsilon(f) \in (0,1)$. Then P is countably generated and projective, but it is not a direct sum of finitely generated projective modules, see [14, §2B]).

7. Minimal approximations

In Section 5, we have proved that almost all cotorsion pairs are complete, so they provide for approximations. In some cases minimal approximations exist, that is, the cotorsion pairs are perfect in the sense of the following definition:

Definition 7.1. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair.

(i) $\mathfrak{C}$ is called *perfect*, provided that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class.

(ii) $\mathfrak{C}$ is called *closed*, provided that $\mathcal{A} = \varinjlim \mathcal{A}$, that is, the class $\mathcal{A}$ is closed under forming direct limits in Mod–R.

The term "perfect" comes from the classical result of Bass characterizing right perfect rings by the property that the cotorsion pair $\mathfrak{P}_0 = (\mathcal{P}_0, \text{Mod–}R)$ is perfect (cf. [2]).

Clearly, any perfect cotorsion pair is complete. The converse fails in general: for example, $\mathfrak{P}_0$ is complete for any ring.

In order to prove the existence of minimal approximations, we will often use the following version of a result due to Enochs and Xu [20, §2.2]:

Theorem 7.2. *Let R be a ring and M be a module. Let $\mathcal{C}$ be a class of modules closed under extensions and direct limits. Assume that M has a special $\mathcal{C}^\perp$-preenvelope ν with* $\operatorname{Coker} \nu \in \mathcal{C}$. *Then M has a $\mathcal{C}^\perp$-envelope.*

Proof. By an ad hoc notation, we will call an exact sequence $0 \to M \to F \to C \to 0$ with $C \in \mathcal{C}$ an *Ext-generator*, provided that for each exact sequence $0 \to M \to F' \to C' \to 0$ with $C' \in \mathcal{C}$ there exist $f \in \operatorname{Hom}_R(F', F)$ and $g \in \operatorname{Hom}_R(C', C)$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. By assumption, there exists an Ext-generator with the middle term $F \in \mathcal{C}^\perp$. The proof is divided into three steps:

Lemma 7.3. *Assume $0 \to M \to F \to C \to 0$ is an Ext-generator. Then there exist an Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

such that $\operatorname{Ker}(f) = \operatorname{Ker}(f'f)$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0.
\end{tikzcd}$$

Proof. Assume that the assertion does not hold. By induction, we will construct a direct system of Ext-generators indexed by ordinals as follows:
First let the second row be the same as the first one, that is, put $F' = F_0 = F$, $C' = C_0 = C$, $f = \mathrm{id}_F$ and $g = \mathrm{id}_C$. Then there exist $F_1 = F''$, $C_1 = C''$, $f_{10} = f'$ and $g_{10} = g'$, such that the diagram above commutes, its rows are Ext-generators and $\operatorname{Ker} f_{10} \supsetneq \operatorname{Ker} f = 0$.

Assume that the Ext-generator $0 \to M \to F_\alpha \to C_\alpha \to 0$ is defined together with $f_{\alpha\beta} \in \operatorname{Hom}_R(F_\beta, F_\alpha)$ and $g_{\alpha\beta} \in \operatorname{Hom}_R(C_\beta, C_\alpha)$ for all $\beta \leq \alpha$. Then there exist $F_{\alpha+1}$, $C_{\alpha+1} \in \mathcal{C}$, $f_{\alpha+1,\alpha}$ and $g_{\alpha+1,\alpha}$, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F_\alpha \arrow[r] \arrow[d, "f_{\alpha+1,\alpha}"] & C_\alpha \arrow[r] \arrow[d, "g_{\alpha+1,\alpha}"] & 0 \\
0 \arrow[r] & M \arrow[r] & F_{\alpha+1} \arrow[r] & C_{\alpha+1} \arrow[r] & 0
\end{tikzcd}$$

commutes, its rows are Ext-generators and $\operatorname{Ker} f_{\alpha+1,0} \supsetneq \operatorname{Ker} f_{\alpha 0}$, where $f_{\alpha+1,\beta} = f_{\alpha+1,\alpha} f_{\alpha\beta}$ and $g_{\alpha+1,\beta} = g_{\alpha+1,\alpha} g_{\alpha\beta}$ for all $\beta \leq \alpha$.

If α is a limit ordinal, put $F_\alpha = \varinjlim_{\beta<\alpha} F_\beta$ and $C_\alpha = \varinjlim_{\beta<\alpha} C_\beta$. Consider the direct limit $0 \to M \to F_\alpha \to C_\alpha \to 0$ of the Ext-generators $0 \to M \to F_\beta \to C_\beta \to 0$, $(\beta < \alpha)$. Since $\mathcal{C}$ is closed under direct limits, we have $C_\alpha \in \mathcal{C}$. Since $0 \to M \to F_\beta \to C_\beta \to 0$ is an Ext-generator for (some) $\beta < \alpha$, also $0 \to M \to F_\alpha \to C_\alpha \to 0$ is an Ext-generator.

Put $f_{\alpha\beta} = \varinjlim_{\beta \leq \beta' < \alpha} f_{\beta'\beta}$ and $g_{\alpha\beta} = \varinjlim_{\beta\leq\beta'<\alpha} g_{\beta'\beta}$ for all $\beta < \alpha$. Then $\operatorname{Ker}(f_{\alpha 0}) \supseteq \operatorname{Ker}(f_{\beta 0})$, and hence $\operatorname{Ker}(f_{\alpha 0}) \supsetneq \operatorname{Ker}(f_{\beta 0})$, for each $\beta < \alpha$.

By induction, for each ordinal α we obtain a strictly increasing chain $(\operatorname{Ker} f_{\beta 0} \mid \beta < \alpha)$, consisting of submodules of F, a contradiction. □

Lemma 7.4. *Assume* $0 \to M \to F \to C \to 0$ *is an Ext-generator. Then there exist an Ext-generator* $0 \to M \to F' \to C' \to 0$ *and a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & F' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

such that $\operatorname{Ker}(f') = 0$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F' \arrow[r] \arrow[d, "f'"] & C' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & M \arrow[r] & F'' \arrow[r] & C'' \arrow[r] & 0.
\end{tikzcd}$$

Proof. By induction on $n < \omega$, we infer from Lemma 7.3 that there is a countable direct system $\mathcal{D}$ of Ext-generators $0 \to M \to F_n \to C_n \to 0$ with homomorphisms $f_{n+1,n} \in \operatorname{Hom}_R(F_n, F_{n+1})$, $g_{n+1,n} \in \operatorname{Hom}_R(C_n, C_{n+1})$, such that the 0-th term of $\mathcal{D}$ is the given Ext-generator $0 \to M \to F \to C \to 0$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_{n+1,n}} & & \downarrow{\scriptstyle g_{n+1,n}} & & \\ 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \end{array}$$

is commutative, and for each commutative diagram whose rows are Ext-generators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \bar f} & & \downarrow{\scriptstyle \bar g} & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar F & \longrightarrow & \bar C & \longrightarrow & 0 \end{array}$$

we have $\mathrm{Ker}(f_{n+1,n}) = \mathrm{Ker}(\bar f f_{n+1,n})$.

Consider the direct limit $0 \to M \to F' \to C' \to 0$ of $\mathcal{D}$, so $F' = \varinjlim_{n<\omega} F_n$ and $C' = \varinjlim_{n<\omega} C_n$. Since $\mathcal{C}$ is closed under direct limits, we have $C' \in \mathcal{C}$, and $0 \to M \to F' \to C' \to 0$ is an Ext-generator. It is easy to check that this generator has the required injectivity property. □

Lemma 7.5. *Let $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ be the Ext-generator constructed in Lemma 7.4. Then $\nu : M \to F'$ is a $\mathcal{C}^{\perp}$-envelope of M.*

Proof. First we prove that in each commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

f' is an automorphism.

Assume this is not true. By induction, we construct a direct system of Ext-generators, $0 \to M \to F_\alpha \to C_\alpha \to 0$, indexed by ordinals, with injective, but not surjective, homomorphisms $f_{\alpha\beta} \in \mathrm{Hom}_R(F_\beta, F_\alpha)$ $(\beta < \alpha)$. In view of Lemma 7.4, we take

$$0 \to M \to F_\alpha \to C_\alpha \to 0 = 0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$$

in case $\alpha = 0$ or α non-limit and $F_\alpha = \varinjlim F_\beta$ and $C_\alpha = \varinjlim C_\beta$, if α is a limit ordinal. Then for each non-limit ordinal α $(\mathrm{Im}\, f_{\alpha\beta} \mid \beta \text{ non-limit}, \beta < \alpha)$ is a strictly increasing sequence of submodules of F', a contradiction.

It remains to prove that $F' \in \mathcal{C}^{\perp}$. Consider an exact sequence $0 \to F' \xrightarrow{\mu} X \to C \to 0$, where $C \in \mathcal{C}$. We will prove that this sequence splits.
Consider the pushout of π and μ:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\nu} & F' & \xrightarrow{\pi} & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mu} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

Since $\mathcal{C}$ is closed under extensions, we have $P \in \mathcal{C}$. Since $0 \to M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \to 0$ is an Ext-generator, we also have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\ & & \Vert & & \mu' \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\nu} & F' & \xrightarrow{\pi} & C' & \longrightarrow & 0. \end{array}$$

By the first part of the proof, $\mu'\mu$ is an automorphism of F'. It follows that $0 \to F' \xrightarrow{\mu} X \to C \to 0$ splits. $\square$

Theorem 7.6. *Let R be a ring, M be a module and $\mathcal{C}$ be a class of modules closed under direct limits. Assume that M has a $\mathcal{C}$-precover. Then M has a $\mathcal{C}$-cover.*

Proof. The proof is by a construction of precovers with additional injectivity properties. The three steps are analogous to Lemmas 7.3 - 7.5. $\square$

Corollary 7.7. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete and closed cotorsion pair. Then $\mathfrak{C}$ is perfect.*

Proof. By Theorems 7.2 and 7.6. $\square$

The case of $n = 0$ in the following Corollary is the Flat Cover Conjecture proved in [5]:

Corollary 7.8. *Let R be a ring and $n \geq 0$. Then there is a perfect cotorsion pair $\mathfrak{C}_n = (\mathcal{F}_n, \mathcal{G}_n)$ where $\mathcal{F}_n$ denotes the class of all modules of flat dimension $\leq n$.*
In particular, the class $\mathcal{F}_0$ of all flat modules is covering.

Proof. Let $\kappa = |R| + \aleph_0$ and let $\mathcal{C}_n$ denote the class of all $\leq \kappa$-presented modules from $\mathcal{F}_n$. Using purification, we see that $\mathcal{F}_n = \mathrm{Filt}(\mathcal{C}_n)$, so $(\mathcal{F}_n)^\perp = (\mathcal{C}_n)^\perp$ by the Eklof Lemma. By Theorem 5.1, $\mathfrak{C}_n$ is complete. Since $\mathcal{F}_n$ is closed under direct limits, Corollary 7.7 applies. $\square$

8. $\mathcal{C}$-socle sequences and $\mathrm{Filt}(\mathcal{C})$-precovers

It is well-known that if R is a right semiartinian ring, then all modules are semiartinian, and each module has a socle sequence. There is obviously no bound on the lengths of transfinite composition series of modules, but the length of the socle sequence of any module M (called the *Loewy length* of M) is bounded, the bound being the Loewy length of R.

Semiartinian modules coincide with the $\mathcal{C}$-filtered modules for $\mathcal{C} = \mathrm{simp}\, R$. Thus a question arises of whether these well-known facts can be extended to arbitrary $\mathcal{C}$-filtered modules. We are going to present a positive answer: if each module in $\mathcal{C}$ is $< \kappa$-presented, then each $\mathcal{C}$-filtered module has a $\mathcal{C}$-socle sequence of length $\leq \kappa$. As a consequence, we will prove that the class $\mathrm{Filt}(\mathcal{C})$ is precovering. These two remarkable results are due to Enochs [8] and Šťovíček [17].

We start with the definition of a $\mathcal{C}$-socle sequence of a module.

Definition 8.1. Let R be a ring, M be a module, and $\mathcal{C}$ be a class of modules. A continuous chain $\mathcal{N} = (N_\beta \mid \beta \leq \tau)$ of submodules of M is called a *$\mathcal{C}$-socle sequence* of M, provided that $N_\tau = M$, and $N_{\beta+1}/N_\beta$ is isomorphic to a direct sum of elements of $\mathcal{C}$ for each $\beta < \tau$. The ordinal τ is called the *length* of the $\mathcal{C}$-socle sequence $\mathcal{N}$.

It is easy to see that a module M possesses a $\mathcal{C}$-socle sequence, if and only if M is $\mathcal{C}$-filtered. In fact, each $\mathcal{C}$-filtration of M is also its $\mathcal{C}$-socle sequence. So unlike socle sequences, the $\mathcal{C}$-socle sequences are not unique in general. But the key property is the same: $\mathcal{C}$-socle sequences are shorter than $\mathcal{C}$-filtrations in general; moreover, if $\mathcal{C}$ consists of modules of bounded presentation, then each $\mathcal{C}$-filtered module possesses a $\mathcal{C}$-socle sequence of bounded length. Such $\mathcal{C}$-socle sequence can be extracted from the family $\mathcal{F}$ constructed in the Hill Lemma:

Theorem 8.2. ($\mathcal{C}$-socle sequence length bound) *Let R be a ring, κ be an infinite regular cardinal, and $\mathcal{C}$ be a class of $< \kappa$-presented modules. Let M be $\mathcal{C}$-filtered module. Then M has a $\mathcal{C}$-socle sequence of length $\leq \kappa$.*

Proof. Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a $\mathcal{C}$-filtration of M, and $(A_\alpha \mid \alpha < \sigma)$ a family of $< \kappa$-generated submodules of M, such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. Let $\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$ be the family of submodules of M from Theorem 2.8.

Let $\alpha < \sigma$. By Proposition 2.6, and by (the proof of) property (H4) in Theorem 2.8, there is a least closed subset S_α of σ, such that S_α has cardinality $< \kappa$, and $\alpha \in S_\alpha \subseteq \alpha + 1$ (that is, α is the greatest element of S_α).

Putting $\sup(\emptyset) = 0$, we define the $\mathcal{C}$-socle level function $\ell : \sigma \to \kappa$ by induction on $\alpha < \sigma$ by the formula $\ell(\alpha) = \sup\{\ell(\gamma) + 1 \mid \alpha \neq \gamma \in S_\alpha\}$.

Notice that $\ell(\alpha) = 0$ is equivalent to $S_\alpha = \{\alpha\}$ and hence to $M_{\alpha+1} = M_\alpha \oplus A_\alpha$.

For each $\beta \leq \kappa$, let $T_\beta = \{\gamma < \sigma \mid \ell(\gamma) < \beta\}$ and $N_\beta = M(T_\beta)$. We will prove that $\mathcal{N} = (N_\beta \mid \beta \leq \kappa)$ is a $\mathcal{C}$-socle sequence of M.

First, we claim that T_β is closed, hence $N_\beta \in \mathcal{F}$, for each $\beta \leq \kappa$. This will follow once we prove that $T_\beta = \bigcup_{\gamma<\sigma,\ell(\gamma)<\beta} S_\gamma$. However, if $\gamma \in T_\beta$ then $\ell(\gamma) < \beta$, and clearly $\gamma \in S_\gamma$. Conversely, assume that $\alpha \in S_\gamma$ for some $\gamma < \sigma, \ell(\gamma) < \beta$. If $\alpha = \gamma$, then $\alpha \in T_\beta$. Otherwise $\alpha < \gamma$, so $\ell(\alpha) + 1 \leq \ell(\gamma) < \beta$ and $\alpha \in T_\beta$, and the claim is proved.

Clearly $\mathcal{N}$ is a continuous chain of submodules of N, and $N = N_\kappa$. It remains to show that for each $\beta < \kappa$, $N_{\beta+1}/N_\beta \cong \bigoplus_{\gamma \in T_{\beta+1} \setminus T_\beta} \bar{M}_\gamma$, where $\bar{M}_\gamma$ is isomorphic to some element of $\mathcal{C}$ for each $\gamma \in T_{\beta+1} \setminus T_\beta$.

We define $\bar{M}_\gamma = (M(T_\beta) + A_\gamma)/M(T_\beta)$. Then

$$\bar{M}_\gamma = M(T_\beta \cup S_\gamma)/M(T_\beta) \cong M(S_\gamma)/(M(S_\gamma) \cap M(T_\beta)) = M(S_\gamma)/M(S_\gamma \cap T_\beta) =$$

$$= M(S_\gamma)/M(S_\gamma \cap \gamma) \cong A_\gamma/(A_\gamma \cap M(S_\gamma \cap \gamma)) = A_\gamma/(A_\gamma \cap M_\gamma) \cong M_{\gamma+1}/M_\gamma,$$

because $\gamma \in S_\gamma$ and S_γ is closed. However, $M_{\gamma+1}/M_\gamma$ is isomorphic to an element of $\mathcal{C}$ as $\mathcal{M}$ is a $\mathcal{C}$-filtration of M.

Clearly, $N_{\beta+1}/N_\beta \cong \sum_{\gamma \in T_{\beta+1} \setminus T_\beta} \bar{M}_\gamma$.

So it remains to prove that $\bar{M}_\gamma \cap \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} \bar{M}_\delta = 0$, or equivalently,

$$(M(T_\beta) + A_\gamma) \cap (M(T_\beta) + \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} A_\delta) = M(T_\beta).$$

We have $M(T_\beta) + A_\gamma = M(T_\beta \cup S_\gamma)$, and $M(T_\beta) + \sum_{\gamma \neq \delta \in T_{\beta+1} \setminus T_\beta} A_\delta = M(T_\beta \cup \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta)$. Moreover,

$$M(T_\beta \cup S_\gamma) \cap M(T_\beta \cup \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta) = M(T_\beta \cup (S_\gamma \cap \bigcup_{\gamma \neq \delta \in T_{\beta+1}} S_\delta)) = M(T_\beta),$$

because by the definition of the function ℓ, $S_\gamma \cap S_\delta \subseteq T_\beta$ for all $\gamma \neq \delta \in T_{\beta+1}$. $\square$

In order to prove that $\mathrm{Filt}(\mathcal{C})$ is a precovering class for each set of modules $\mathcal{C}$, we will need some preparation.

We start with two lemmas of independent interest. The first one deals with extensions of big direct sums by small modules.

Lemma 8.3. *Let κ and λ be infinite cardinals, and R be a λ-noetherian ring. Let $\mathcal{C}$ be a class of $\leq \kappa$-presented modules, M be a $\leq \lambda$-generated module, and $\tau = \kappa + \lambda$.*

Then for each short exact sequence $0 \to \bigoplus_{i \in I} C_i \subseteq N \xrightarrow{\pi} M \to 0$ with $C_i \in \mathcal{C}$ for all $i \in I$, there exists a subset $J \subseteq I$ of cardinality $\leq \lambda$, and a $\leq \tau$-generated submodule P of N, such that $\bigoplus_{j \in J} C_j \subseteq P$ and $N = P \oplus \bigoplus_{i \in I \setminus J} C_i$.

Proof. By assumption, there is a subset $\{n_\alpha \mid \alpha < \lambda\}$ of N, such that $\pi(N') = M$, where $N' = \sum_{\alpha < \lambda} n_\alpha R$.

Since R is λ-noetherian, the module $N' \cap \bigoplus_{i \in I} C_i$ is $\leq \lambda$-generated. So there is a subset $J \subseteq I$ of cardinality $\leq \lambda$, such that $N' \cap \bigoplus_{i \in I} C_i \subseteq \bigoplus_{j \in J} C_j$.

Let $P = N' + \bigoplus_{j \in J} C_j$. Then P is $\leq \tau$-generated, and $P \cap \bigoplus_{i \in I \setminus J} C_i = 0$, because $N' \cap \bigoplus_{i \in I} C_i \subseteq \bigoplus_{j \in J} C_j$. Finally, $P + \bigoplus_{i \in I \setminus J} C_i = N$, because $\pi(N') = M$. $\square$

Put in other terms, Lemma 8.3 says that the extension

$$0 \to \bigoplus_{i \in I} C_i \subseteq N \xrightarrow{\pi} M \to 0$$

decomposes into the direct sum of the extensions $0 \to \bigoplus_{j \in J} C_j \subseteq P \xrightarrow{\pi \restriction P} M \to 0$ and $0 \to \bigoplus_{i \in I \setminus J} C_i \to \bigoplus_{i \in I \setminus J} C_i \to 0 \to 0$.

In the next lemma, the roles of M and $\bigoplus_{i \in I} C_i$ are swapped, so we are concerned with extensions of small modules by big direct sums.

Lemma 8.4. *Let R be a ring, and κ and λ be infinite cardinals $\geq |R|$. Let $\mathcal{C}$ be a class of $\leq \kappa$-presented modules, M be a module with $|M| \leq \lambda$, and $\tau = \lambda^\kappa + \rho$, where $\rho = \sup_{C \in \mathcal{C}, M' \subseteq M} \left|\operatorname{Ext}^1_R(C, M')\right|$.*

Then, for each short exact sequence $0 \to M \subseteq N \xrightarrow{\pi} \bigoplus_{i \in I} C_i \to 0$ with $C_i \in \mathcal{C}$ for all $i \in I$, there exists a subset $J \subseteq I$ of cardinality $\leq \tau$, such that $P = \pi^{-1}(\bigoplus_{j \in J} C_j)$ is a direct summand of N containing M, with a complement P' isomorphic to $\bigoplus_{i \in I \setminus J} C_i$.

Proof. By the assumption, $\mathcal{C}$ has a representative set of elements $\mathcal{S}$ of cardinality $\leq 2^\kappa$.

For each $i \in I$ and $c \in C_i$, we fix $n_{c,i} \in N$, such that $\pi(n_{c,i}) = c$. Let $N_i = \sum_{c \in C_i} n_{c,i} R$. Then $M + N_i = \pi^{-1}(C_i)$. Let $\sim$ be the equivalence relation on I defined by $i \sim j$, whenever $N_i \cap M = N_j \cap M$ and there is an isomorphism $\varphi_{ij} : N_j \to N_i$, such that $\varphi_{ij} \restriction N_j \cap M = \mathrm{id}$.

Let $J \subseteq I$ be a set of representatives for $\sim$ and let $j \in J$. Then there is a unique $C \in \mathcal{S}$, such that there exists an isomorphism $C_j \xrightarrow{\psi_j} C$.

Let $i \sim j$. Then there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_j \cap M \arrow[r] \arrow[d, equal] & N_j \arrow[r, "\pi \restriction N_j"] \arrow[d, "\varphi_{ij}"'] & C_j \arrow[r] \arrow[d, "\bar{\varphi}_{ij}"'] & 0 \\
0 \arrow[r] & N_i \cap M \arrow[r] & N_i \arrow[r, "\pi \restriction N_i"] & C_i \arrow[r] & 0,
\end{tikzcd}$$

where $\bar{\varphi}_{ij}$ is an isomorphism. Let $\psi_i = \psi_j(\bar{\varphi}_{ij})^{-1}$. Then also the following diagram is commutative and has exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_j \cap M & \longrightarrow & N_j & \xrightarrow{\psi_j \pi \restriction N_j} & C & \longrightarrow & 0 \\ & & \| & & \varphi_{ij} \big\downarrow & & \| & & \\ 0 & \longrightarrow & N_i \cap M & \longrightarrow & N_i & \xrightarrow{\psi_i \pi \restriction N_i} & C & \longrightarrow & 0, \end{array}$$

that is, the upper and lower rows of the diagram are equivalent extensions of $N_j \cap M$ by C.

We claim that $|J| \leq \tau$. Indeed, for $i \in I$, let $t(i) = (N_i \cap M, C, \bar{\mathcal{E}}_i)$, where C is the unique element of $\mathcal{S}$ satisfying $C \cong C_i = (N_i + M)/M$, and $\bar{\mathcal{E}}_i \in \operatorname{Ext}^1_R(C, N_i \cap M)$ is the equivalence class of the extension $0 \to N_i \cap M \to N_i \stackrel{\psi_i \pi \restriction N_i}{\to} C \to 0$. By the above, if $i \in I$ and $j \in J$, then $i \sim j$, if and only if $t(i) = t(j)$. Moreover, $|N_i \cap M| \leq \kappa$ for all $i \in I$, and the claim follows by the definition of τ.

Let $P = M + \sum_{j \in J} N_j$, and $P' = \sum_{j \in J} \sum_{i \sim j, i \neq j} \sum_{c \in C_j} (\mathrm{id}_{N_j} - \varphi_{ij})(n_{c,j})R$. Then $P = \pi^{-1}(\bigoplus_{j \in J} C_j)$, and $P + P' \supseteq N_i$ for each $i \in I \setminus J$, because φ_{ij} is surjective, so $P + P' = N$.

It remains to prove that $P \cap P' = 0$. If $x \in P'$, then x has the form

$$x = \sum_{j \in J} \sum_{i \sim j, i \neq j} \sum_{c \in C_j} (\mathrm{id}_{N_j} - \varphi_{ij})(n_{c,j} r_{c,i,j})$$

and $\pi(x) = \sum_{j \in J} \sum_{i \sim j, i \neq j} \sum_{c \in C_j} (c - \bar{\varphi}_{ij}(c)) r_{c,i,j}$. If, moreover, $x \in P$, then $\pi(x) \in \bigoplus_{j \in J} C_j$, and hence

$$\sum_{j \in J} \sum_{i \sim j, i \neq j} \sum_{c \in C_j} \bar{\varphi}_{ij}(c) r_{c,i,j} = \sum_{j \in J} \sum_{i \sim j, i \neq j} \bar{\varphi}_{ij}(\sum_c c.r_{c,i,j}) = 0.$$

Since $\bar{\varphi}_{ij}$ is an isomorphism, we infer that $\pi(\sum_{c \in C_j} n_{c,j} r_{c,i,j}) = \sum_{c \in C_j} c.r_{c,i,j} = 0$, and $\sum_{c \in C_j} n_{c,j} r_{c,i,j} \in M \cap N_j$ for all $i \sim j$ and $i \neq j$. We conclude that

$$x = \sum_{j \in J} \sum_{i \sim j, i \neq j} (\mathrm{id}_{N_j} - \varphi_{ij})(\sum_{c \in C_j} n_{c,j} r_{c,i,j}) = 0,$$

because $\varphi_{ij} \restriction (N_j \cap M) = \mathrm{id}_{N_j}$ for all $i \sim j$ and $i \neq j$. $\square$

Lemma 8.4 just says that the extension $0 \to M \subseteq N \xrightarrow{\pi} \bigoplus_{i \in I} C_i \to 0$ decomposes into the direct sum of the extensions $0 \to M \subseteq P \stackrel{\pi \restriction P}{\to} \bigoplus_{j \in J} C_j \to 0$ and $0 \to 0 \to P' \to \bigoplus_{i \in I \setminus J} C_i \to 0$.

The following lemma builds $\mathrm{Filt}(\mathcal{C})$-approximations gradually, by induction on the length of a $\mathcal{C}$-socle sequence.

Lemma 8.5. *Let κ be an infinite cardinal and $\mathcal{C}$ a class of $\leq \kappa$-presented modules. Then for each module T there is a sequence $((S_\alpha, f_\alpha) \mid \alpha \leq \kappa^+)$ with the following properties:*

(a) *$(S_\beta \mid \beta \leq \gamma)$ is a $\mathcal{C}$-socle sequence of S_γ for each $\gamma \leq \kappa^+$.*

(b) *$f_\gamma \in \operatorname{Hom}_R(S_\gamma, T)$ for each $\gamma \leq \kappa^+$. The morphisms f_γ ($\gamma \leq \kappa^+$) are compatible, that is, $f_\gamma \restriction S_\beta = f_\beta$ for all $\beta < \gamma \leq \kappa^+$.*

(c) *Let $\gamma \leq \kappa^+$, X be a module possessing a $\mathcal{C}$-socle sequence $(X_\beta \mid \beta \leq \gamma)$, and $h_\gamma \in \operatorname{Hom}_R(X, T)$. Then there exists $g_\gamma \in \operatorname{Hom}_R(X, S_\gamma)$ such that $g_\gamma \restriction X_\beta \in \operatorname{Hom}_R(X_\beta, S_\beta)$ for each $\beta < \gamma$, and $h_\gamma = f_\gamma g_\gamma$.*

Moreover, if $\gamma \leq \kappa^+$ is a non-limit ordinal and there exists a homomorphism $g \in \operatorname{Hom}_R(X_{\gamma-1}, S_{\gamma-1})$ such that $g \restriction X_\beta \in \operatorname{Hom}_R(X_\beta, S_\beta)$ for all $\beta < \gamma$, and $h_\gamma \restriction X_{\gamma-1} = f_{\gamma-1}(g \restriction S_{\gamma-1})$, then the map g_γ above can be chosen so that $g_\gamma \restriction X_{\gamma-1} = g$.

Proof. Let T be an arbitrary module. By induction on $\alpha \leq \kappa^+$, we will construct a sequence $((S_\beta, f_\beta) \mid \beta \leq \alpha)$ so that conditions (a)-(c) hold for each $\gamma \leq \alpha$. For $\alpha = 0$ we let $S_0 = 0$ and $f_0 = 0$.

Assume $\alpha \geq 0$ and the sequence $((S_\beta, f_\beta) \mid \beta \leq \alpha)$ has been constructed. We have to define the pair $(S_{\alpha+1}, f_{\alpha+1})$ so that $S_\alpha \subseteq S_{\alpha+1}$, $S_{\alpha+1}/S_\alpha$ is a direct sum of copies of elements of $\mathcal{C}$, $f_{\alpha+1} \in \operatorname{Hom}_R(S_{\alpha+1}, T)$, $f_{\alpha+1} \restriction S_\alpha = f_\alpha$, and condition (c) holds for $\gamma = \alpha + 1$.

Let $\mathcal{S}$ be a representative set of elements of $\mathcal{C}$, $E = \bigoplus_{u \in \operatorname{Hom}_R(C_u, T), C_u \in \mathcal{S}} C_u$, and $v \in \operatorname{Hom}_R(E, T)$ be the canonical map (i.e., $v \restriction C_u = u$). Then each homomorphism from a direct sum of copies of elements of $\mathcal{C}$ to T factors through v.

Let $\tau = \lambda^\kappa + \rho$, where $\lambda = \aleph_0 + |S_\alpha| + R$, and $\rho = \sup_{C \in \mathcal{S}, M' \subseteq S_\alpha} \left|\operatorname{Ext}^1_R(C, M')\right|$. Also, let $\mathcal{E}_\gamma : 0 \to S_\alpha \subseteq P_\gamma \to D_\gamma \to 0$ ($\gamma < \rho$) be a representative set of all equivalence classes of extensions of S_α by direct sums of $\leq \tau$ elements of $\mathcal{S}$. For each $\gamma < \rho$, we define $H_\gamma = \{f \in \operatorname{Hom}_R(P_\gamma, T) \mid f \restriction S_\alpha = f_\alpha\}$. Consider the pushout

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{\gamma<\rho} S_\alpha^{(H_\gamma)} & \xrightarrow{\subseteq} & \bigoplus_{\gamma<\rho} P_\gamma^{(H_\gamma)} & \longrightarrow & \bigoplus_{\gamma<\rho} D_\gamma^{(H_\gamma)} & \longrightarrow & 0 \\
 & & \Phi\big\downarrow & & \eta\big\downarrow & & \big\| & & \\
0 & \longrightarrow & S_\alpha & \xrightarrow{\subseteq} & S'_\alpha & \longrightarrow & \bigoplus_{\gamma<\rho} D_\gamma^{(H_\gamma)} & \longrightarrow & 0,
\end{array}$$

where $\Phi : \bigoplus_{\gamma<\rho} S_\alpha^{(H_\gamma)} \to S_\alpha$ is the summation map. Let $c : \bigoplus_{\gamma<\rho} P_\gamma^{(H_\gamma)} \to T$ be the canonical map. Then $c \restriction \bigoplus_{\gamma<\rho} S_\alpha^{(H_\gamma)} = f_\alpha \Phi$, so the pushout property yields $f'_\alpha \in \operatorname{Hom}_R(S'_\alpha, T)$ such that $f'_\alpha \restriction S_\alpha = f_\alpha$ and $f'_\alpha \eta = c$.

We define $S_{\alpha+1} = S'_\alpha \oplus E$, and $f_{\alpha+1} = f'_\alpha \oplus v$. Clearly, $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\gamma<\rho} D_\gamma^{(H_\gamma)} \oplus E$, and $f_{\alpha+1} \restriction S_\alpha = f'_\alpha \restriction S_\alpha = f_\alpha$.

Now, let X be an arbitrary module possessing a $\mathcal{C}$-socle sequence $(X_\beta \mid \beta \leq \alpha{+}1)$, and $h_{\alpha+1} \in \operatorname{Hom}_R(X, T)$.

There is a short exact sequence $0 \to X_\alpha \subseteq X \to D \to 0$, where D is a direct sum of modules from $\mathcal{S}$. By the inductive premise for (c), there exists $g_\alpha \in \operatorname{Hom}_R(X_\alpha, S_\alpha)$ such that $g_\alpha \restriction X_\beta \in \operatorname{Hom}_R(X_\beta, S_\beta)$ for each $\beta \leq \alpha$, and $h_{\alpha+1} \restriction X_\alpha = f_\alpha g_\alpha$. Moreover, if g is given as in the second paragraph of (c) for $\gamma = \alpha + 1$, then we can choose $g_\alpha = g$. Consider the pushout

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_\alpha & \xrightarrow{\subseteq} & X & \longrightarrow & D & \longrightarrow & 0 \\
 & & g_\alpha\big\downarrow & & z_\alpha\big\downarrow & & \big\| & & \\
0 & \longrightarrow & S_\alpha & \xrightarrow{\subseteq} & N & \longrightarrow & D & \longrightarrow & 0.
\end{array}$$

By the pushout property, there is $y_\alpha \in \operatorname{Hom}_R(N, T)$ such that $y_\alpha \restriction S_\alpha = f_\alpha$ and $h_{\alpha+1} = y_\alpha z_\alpha$.

We can apply Lemma 8.4 to the bottom short exact sequence (so $M = S_\alpha$, and λ and τ are as above), and obtain a decomposition $N = P \oplus P'$, where $S_\alpha \subseteq P$, $|P| \leq \tau$, and both P/S_α and P' are isomorphic to direct sums of elements of $\mathcal{S}$.

It remains to construct $g_{\alpha+1} \in \operatorname{Hom}_R(X, S_{\alpha+1})$ so that $g_{\alpha+1} \restriction X_\alpha = g_\alpha$ and $h_{\alpha+1} = f_{\alpha+1} g_{\alpha+1}$.

Since v is the canonical map, $y_\alpha \restriction P'$ factors through it, that is, there exists $w \in \operatorname{Hom}_R(P', E)$ such that $y_\alpha \restriction P' = vw$.

In order to factor $y_\alpha \restriction P$, we note that $y_\alpha \restriction S_\alpha = f_\alpha$, and let δ be the index corresponding to the extension $0 \to S_\alpha \to P \to P/S_\alpha \to 0$ and to $y_\alpha \restriction P \in \operatorname{Hom}_R(P, T)$. Let $\mu_\delta : P \to \bigoplus_{\gamma<\rho} P_\gamma^{(H_\gamma)}$ be the corresponding split embedding. We define $g_{\alpha+1} = (\eta\mu_\delta \oplus w) z_\alpha \in \operatorname{Hom}_R(X, S_{\alpha+1})$.

Then $g_{\alpha+1} \restriction X_\alpha = \Phi\mu_\delta g_\alpha = g_\alpha$. Moreover,

$$f_{\alpha+1}g_{\alpha+1} = (f'_\alpha \oplus v)(\eta\mu_\delta \oplus w)z_\alpha = (c\mu_\delta \oplus y_\alpha \restriction P')z_\alpha = y_\alpha z_\alpha = h_{\alpha+1}.$$

Finally, if α is a limit ordinal, we define $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Then (a) and (b) clearly hold also for $\gamma = \alpha$.

Assume X and h_α are given as in (c) (for $\gamma = \alpha$). Using the second paragraph of (c) by induction on $\gamma < \alpha$, we can define a compatible system of morphisms g_β ($\beta < \alpha$) such that $g_\beta \in \operatorname{Hom}_R(X_\beta, S_\beta)$ and $h_\alpha \restriction X_\beta = f_\beta g_\beta$ for each $\beta < \alpha$. Let $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. Then $h_\alpha = f_\alpha g_\alpha$, so (c) holds also for $\gamma = \alpha$. □

Theorem 8.6. (Filt($\mathcal{C}$) is a precovering class) *Let R be a ring, κ be an infinite cardinal, and $\mathcal{C}$ be a class of $\leq \kappa$-presented modules. Let $\alpha \leq \kappa^+$, and let* $\operatorname{Soc}_\alpha(\mathcal{C})$ *denote the class of all modules possessing a $\mathcal{C}$-socle sequence of length $\leq \alpha$.*

Then $\operatorname{Soc}_\alpha(\mathcal{C})$ *is a precovering class, and so are the classes* $\operatorname{Filt}(\mathcal{C})$ *and* $\mathcal{D} = \operatorname{Add}(\operatorname{Filt}(\mathcal{C}))$.

Proof. Let T be a module, and S_α and f_α be as in Lemma 8.5. Consider $X \in \operatorname{Soc}_\alpha(\mathcal{C})$ and $h \in \operatorname{Hom}_R(X, T)$. Let $(X_\beta \mid \beta \leq \alpha)$ be a $\mathcal{C}$-socle sequence of X. By Lemma 8.5(c), there exists $g_\alpha \in \operatorname{Hom}_R(X, S_\alpha)$ such that $h = f_\alpha g_\alpha$. So f_α is a $\operatorname{Soc}_\alpha(\mathcal{C})$-precover of T.

In particular, for $\alpha = \kappa^+$ we infer from Theorem 8.2 that the class $\operatorname{Filt}(\mathcal{C}) = \operatorname{Soc}_{\kappa^+}(\mathcal{C})$ is precovering. The final claim follows from the easy observation that each $\operatorname{Filt}(\mathcal{C})$-precover is also a $\mathcal{D}$-precover. □

Remark 8.7. If, moreover, $R \in \mathcal{C}$, then the class $\mathcal{D}$ is even special precovering by Theorem 5.1 and Corollary 5.3. Also, by Theorem 6.2, if κ is a regular infinite cardinal, $\mathcal{C}$ consists of $< \kappa$-presented modules, and $(\mathcal{A}, \mathcal{B})$ is the cotorsion pair generated by $\mathcal{C}$, then $\operatorname{Filt}(\mathcal{A}^{<\kappa}) = \mathcal{A}$ is a special precovering class.

Assume that $\mathcal{C}$ is a precovering class such that each $\mathcal{C}$-precover is surjective (e.g., $\mathcal{C}$ contains all free modules). Then by an iteration of precovers, of a module M, of the kernel of a $\mathcal{C}$-precover of M, etc., we obtain a $\mathcal{C}$*-resolution* of M. In the classic particular case when $\mathcal{C}$ is the class of all projective modules, this is just the projective resolution of M.

As in the classic case, the general $\mathcal{C}$-resolutions are not unique, but one can use them to define uniquely the relative cohomology groups, and derive the long exact sequence for them as in the classic case (see e.g. [9]).

In fact, most classes used to define relative cohomology groups are even deconstructible:

Definition 8.8. A class of modules $\mathcal{C}$ is *deconstructible* provided there exists a cardinal κ such that $\mathcal{C} = \operatorname{Filt}(\mathcal{D})$, where $\mathcal{D}$ is the class of all $\leq \kappa$-presented modules from $\mathcal{C}$.

Corollary 8.9. *Each deconstructible class of modules is precovering.*

Remark 8.10. For example, for each $n \geq 0$, the class $\mathcal{P}_n$ of all modules of projective dimension $\leq n$, as well as the class $\mathcal{F}_n$ of all modules of flat dimension $\leq n$, are deconstructible (in fact, all these classes are *special precovering*, cf. Corollary 7.8 and [10, §8.1]). However, the class of all $\aleph_1$-projective modules is not deconstructible for any non-right perfect ring, [11].

Open problem. Let R be a ring and $\mathcal{C}$ be a covering class of modules. Is $\mathcal{C}$ closed under direct limits? In particular, is every perfect cotorsion pair closed? (Cf. Corollary 7.7.)

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