Tilting theory for commutative rings

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Tilting module = n–tilting module for some $n < \omega$. The tilting class induced by T is $T^{\perp} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i_R(T, M) = 0 \text{ for all } i \geq 1\}$.

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A tilting module T is good if (T3) holds with $\mathrm{Add}\, T$ replaced by $\mathrm{add}\, T$.

The tilting modules T and T' are equivalent if $T^{\perp} = (T')^{\perp}$.

Each tilting module is equivalent to a good one.

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Let T be a classical n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} Ker(Ext^j_R(T,-)) \underset{Tor^i_S(-,T)}{\overset{Ext^i_R(T,-)}{\rightleftarrows}} \bigcap_{j \leq n, j \neq i} Ker(Tor^S_j(-,T))$$

where $S = End_R(T)$.

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General i-tilting theorem

Let R be a ring and T be a good n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j\leq n, j\neq i} \operatorname{Ker}(\operatorname{Ext}^j_R(T,-)) \underset{\operatorname{Tor}^i_S(-,T)}{\overset{\operatorname{Ext}^i_R(T,-)}{\rightleftarrows}} \bigcap_{j\leq n, j\neq i} \operatorname{Ker}(\operatorname{Tor}^S_j(-,T)) \cap \mathcal{E}_\perp$$

where $S = \operatorname{End}_R(T)$, $\mathcal{E}_\perp = \{X \in \mathcal{D}(S) \mid \operatorname{Hom}_{\mathcal{D}(S)}(\mathcal{E}, X) = 0\}$, and $\mathcal{E}$ is the kernel of the total left derived functor $\mathbb{L}(- \otimes_S T)$.

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Then $\mathcal{T}$ is n–tilting, iff there is a set $\mathcal{S}$ consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $\mathcal{T} = \mathcal{S}^\perp$ (i.e., $\mathcal{T}$ is of finite type).

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In particular, each tilting class is definable, i.e., closed under direct products, direct limits, and pure submodules.

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Given a $P \subseteq \mathrm{mSpec}(R)$, the corresponding tilting class is
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This class is induced by the Bass tilting module, i.e., the tilting module $T_P = R_P \oplus R_P/R$ where $R_P = \bigcap_{\mathfrak{q} \in \mathrm{mSpec}(R) \setminus P} R_{\mathfrak{q}}$ and $R_{\mathfrak{q}}$ denotes the localization of R at $\mathfrak{q}$.

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Remark: These are exactly the the special preenveloping torsion classes in $\mathrm{Mod}\text{–}R$.

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This class is induced by the tilting module $T_P = R_P \oplus \bigoplus_{\mathfrak{p} \in P} E(R/\mathfrak{p})$ where R_P is the subring of $Q_{cl}(R)$ containing R and satisfying $R_P/R \cong \bigoplus_{\mathfrak{p} \in P} E(R/\mathfrak{p})$.

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Given $P \subseteq P_1$, the corresponding tilting class is

$$\mathcal{T}_P = \{M \in \text{Mod-}R \mid \operatorname{Ext}^1_R(E(R/\mathfrak{p}), M) = 0 \text{ for all } \mathfrak{p} \in P\}.$$

This class is induced by the tilting module $T_P = R_P \oplus \bigoplus_{\mathfrak{p} \in P} E(R/\mathfrak{p})$ where R_P is the subring of $Q_{cl}(R)$ containing R and satisfying $R_P/R \cong \bigoplus_{\mathfrak{p} \in P} E(R/\mathfrak{p})$. The T_P is called the Bass tilting module.

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Moreover, $\mathcal{T}_P = \mathcal{S}_P^{\perp}$, where $\mathcal{S}_P = \{F_{\mathfrak{p}} \mid \mathfrak{p} \in P\}$, and $F_{\mathfrak{p}}$ is the Auslander–Buchweitz approximation of $R/\mathfrak{p}$.

Tilting for regular rings of Krull dimension two

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3. two exceptional tilting modules T_e and T_f.

Example

The tilting class $\mathcal{I}_1$ is induced by an exceptional tilting module T_e such that T_e is countably generated, torsionfree, and $\operatorname{pd} T_e = 1$.

The dual setting

The dual setting

Definition

Let R be a ring and $n < \omega$. A left R–module C is n–cotilting provided

(C1) $\mathrm{id}_R(C) \leq n$.

(C2) $\mathrm{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i$ and all cardinals κ.

(C3) There is an injective cogenerator W and a long exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to W \to 0$, with $C_i \in \mathrm{Prod}\, C$.

The class ${}^\perp C = \{M \in R\text{–Mod} \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } i \geq 1\}$ is the cotilting class induced by C.

The cotilting modules C and C' are equivalent if ${}^\perp C = {}^\perp C'$.

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Let R be a ring, $n \geq 0$, and T be an n–tilting right R–module. Then the dual module $C = T^* = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is an n–cotilting left R–module.

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The tilting right R–modules T and T' are equivalent, iff the dual modules T^* and $(T')^*$ are equivalent cotilting left R–modules.

Moreover, if $\mathcal{S}$ is a set consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $T^\perp = \mathcal{S}^\perp$ is the tilting class induced by T, then

$$^\perp T^* = \mathcal{S}^\intercal = \{N \in R\text{–}\mathrm{Mod} \mid \mathrm{Tor}_i^R(S, N) = 0 \text{ for all } i \geq 1 \text{ and } S \in \mathcal{S}\}$$

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The map $T \mapsto T^*$ induces a bijection between equivalence classes of tilting modules on the one hand, and equivalence classes of cotilting modules of cofinite type on the other hand.

Similarly, the maps

$$\mathcal{T} \mapsto ({}^\perp\mathcal{T} \cap \text{mod-}R)^\intercal$$

and

$$\mathcal{C} \mapsto ({}^\intercal\mathcal{C} \cap \text{mod-}R)^\perp$$

provide for a 1–1 correspondence between tilting classes, and cotilting classes of cofinite type.

Valuation domains and cofinite type

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Theorem

Let R be a valuation domain. Then all cotilting classes are of cofinite type, iff R is strongly discrete (that is, R has no non–zero idempotent prime ideals).

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Example

Let R be a maximal valuation domain with an idempotent maximal ideal $\mathfrak{m}$. Then the class of all modules M whose torsion part is annihilated by $\mathfrak{m}$ is 1–cotilting, but not of cofinite type.

The role of associated primes in the noetherian setting

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A subset $P \subseteq \mathrm{Spec}(R)$ is closed under generalization provided that $(P, \subseteq)$ is a lower subset in $(\mathrm{Spec}(R), \subseteq)$.

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Theorem (The structure of 1–cotilting classes)

Let R be a commutative noetherian ring. Then there is a 1–1 correspondence between

1. *the 1–cotilting classes $\mathcal{C}$ in Mod-R, and*
2. *the subsets P of $\mathrm{Spec}(R)$ containing $\mathrm{Ass}(R)$ and closed under generalization.*

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It is given by the inverse assignments $\mathcal{C} \mapsto \mathrm{Ass}(\mathcal{C})$ and $P \mapsto \{M \in \text{Mod-}R \mid \mathrm{Ass}(M) \subseteq P\}$.

The Auslander–Bridger transpose

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Let $C \in \mathrm{mod}\text{–}R$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ be a projective presentation of C. The transpose of C, denoted by $\mathrm{Tr}(C)$, is the cokernel of f^+, where $(-)^+ = \mathrm{Hom}_R(-, R)$.

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$\mathrm{Tr}(C)$ is uniquely determined up to adding or splitting off a projective summand.

Lemma

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be such that $\mathrm{Ass}(R) \cap V(\mathfrak{p}) = \emptyset$. Then

(i) $pd_R(\mathrm{Tr}(R/\mathfrak{p})) \leq 1$;

(ii) *$Hom_R(R/\mathfrak{p}, -)$ and $Tor_1^R(\mathrm{Tr}(R/\mathfrak{p}), -)$ are isomorphic functors.*

A classification of 1–tilting classes

A classification of 1–tilting classes

Corollary

Let R be a commutative noetherian ring. Then all 1–cotilting classes are of cofinite type, so there is a bijection between 1–tilting classes and the subsets P of $\mathrm{Spec}(R)$ containing $\mathrm{Ass}(R)$ and closed under generalization.

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Corollary

Let R be a commutative noetherian ring. Then all 1–cotilting classes are of cofinite type, so there is a bijection between 1–tilting classes and the subsets P of $\mathrm{Spec}(R)$ containing $\mathrm{Ass}(R)$ and closed under generalization. For such P, the corresponding 1–tilting class is

$$\mathcal{T} = \bigcap_{\mathfrak{q} \in \mathrm{Spec}(R) \setminus P} \mathrm{Tr}(R/\mathfrak{q})^{\perp}.$$

Characteristic sequences

Characteristic sequences

Definition

Let R be a commutative noetherian ring. A sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ of subsets of $\mathrm{Spec}(R)$ is called characteristic provided that

(i) P_i is closed under generalization for all $i < n$,

(ii) $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1}$, and

(iii) $\mathrm{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$.

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For each characteristic sequence $\mathcal{P}$, we define the class of modules

$$\mathcal{C}_{\mathcal{P}} = \{M \in \text{Mod-}R \mid \mathrm{Ass}(\Omega^{-i}(M)) \subseteq P_i \text{ for all } i < n\}$$

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$$\mathcal{C} \mapsto (\mathrm{Ass}(\mathcal{C}_0), \dots, \mathrm{Ass}(\mathcal{C}_{n-1}))$$

and

$$\mathcal{P} = (P_0, \dots, P_{n-1}) \mapsto \mathcal{C}_{\mathcal{P}}$$

are inverse bijections.

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Lemma

Let R be a ring and C be an n–cotilting module with the induced class $\mathcal{C}$. For each $i \leq n$, let $\mathcal{C}_i = {}^\perp\Omega^{-i}(C)$.

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Let R be a ring and C be an n–cotilting module with the induced class $\mathcal{C}$. For each $i \leq n$, let $\mathcal{C}_i = {}^\perp\Omega^{-i}(C)$. Then $\mathcal{C}_i$ is an $(n-i)$–cotilting class.

The transpose revisited

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Lemma

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ *and* $n \geq 1$ *such that* $\mathrm{Ass}(\Omega^{-i}(R)) \cap V(\mathfrak{p}) = \emptyset$ *for each* $i < n$. *Then*

(i) $pd_R(\mathrm{Tr}(R/\mathfrak{p})) \leq n$.

(ii) $Ext_R^{n-1}(R/\mathfrak{p}, -)$ *and* $Tor_1^R(\mathrm{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), -)$ *are isomorphic functors.*

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Complete classification for commutative noetherian rings

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Theorem

Let $n \geq 1$. Then there are bijections between:

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A characteristic sequence $(P_0, \dots, P_{n-1})$ *corresponds to the n–tilting class*

$$\begin{aligned}\mathcal{T} =& \{M \in \mathrm{Mod}\text{–}R \mid Tor_i^R(R/\mathfrak{p}, M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\} = \\ & \{M \in \mathrm{Mod}\text{–}R \mid Ext_R^1(\mathrm{Tr}(\Omega^{(i)}(R/\mathfrak{p})), M) = 0 \,\forall i < n \,\forall \mathfrak{p} \notin P_i\},\end{aligned}$$

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and the n–cotilting class

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Lemma (uniqueness)

If C and C' are minimal cotilting modules such that C is equivalent to C', then $C \cong C'$.

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Example

Let R be a commutative noetherian ring and $C = \bigoplus_{\mathfrak{m} \in \mathrm{mSpec}(R)} E(R/\mathfrak{m})$. Then C is a minimal 0-cotilting module (= minimal injective cogenerator).

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For each $i < n$, let $\mathcal{I}(P_i)$ be the class of all injective modules I with $\mathrm{Ass}(I) \subseteq P_i$.

For each $i < n$ and each non-empty subset $S \subseteq P_i \setminus P_{i-1}$, let $E_S = \bigoplus_{\mathfrak{p} \in S} E(R/\mathfrak{p})$ and consider the long exact sequence

$$0 \to C_S \to E_0 \xrightarrow{\varphi_0} E_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{i-2}} E_{i-1} \xrightarrow{\varphi_{i-1}} E_S \to 0$$

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such that φ_{i-1} is a $\mathcal{I}(P_{i-1})$-cover of E_S, and for each $0 < j < i-1$, $\varphi_j = \mu_j \circ \psi_j$, where μ_j is the inclusion of $K_j = \mathrm{Ker}(\varphi_{j+1})$ into E_{j+1}, and $\psi_j : E_j \to K_j$ is a $\mathcal{I}(P_j)$-cover.

The structure of minimal cotilting modules

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Let R be a commutative noetherian ring. Let $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a characteristic sequence and $\mathcal{C}$ be the corresponding n-cotilting class.

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Moreover, $C \cong C_{S_0} \oplus \cdots \oplus C_{S_n}$ where S_i is the set of all maximal elements in $P_i \setminus P_{i-1}$, for all $i \leq n$.

Cotilting and colocalization

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Definition

Let R be a commutative ring, M an R-module, and $\mathfrak{m} \in \mathrm{mSpec}(R)$. Denote by $M^{\mathfrak{m}}$ the $R_{\mathfrak{m}}$-module $\mathrm{Hom}_R(R_{\mathfrak{m}}, M)$; it is called the colocalization of M at $\mathfrak{m}$.

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Theorem

Let R be a commutative noetherian ring, $n < \omega$, and C be an n-cotilting R-module.

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Theorem

Let R be a commutative noetherian ring, $n < \omega$, and C be an n-cotilting R-module.
Then for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $C^{\mathfrak{m}}$ is an n-cotilting $R_{\mathfrak{m}}$-module, and $D = \prod_{\mathfrak{m} \in \mathrm{mSpec}(R)} C^{\mathfrak{m}}$ is an n-cotilting R-module equivalent to C.

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Moreover, $(C^{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of n-cotilting modules, and cotilting R-modules correspond 1-1 to such compatible families.

Tilting and localization

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Theorem

Let R be a commutative ring, $n < \omega$, and T be an n-tilting R-module. Then for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $T_{\mathfrak{m}}$ is an n-tilting $R_{\mathfrak{m}}$-module.

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Remark

If R is moreover noetherian, then $(T_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of n-tilting modules. Tilting R-modules correspond 1-1 to such compatible families.

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Remark

If R is moreover noetherian, then $(T_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of n-tilting modules. Tilting R-modules correspond 1-1 to such compatible families.
However, there is no simple way to recover T from the compatible family $(T_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$. !!!

A research outlook

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The APD and Prüfer cases are done.

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