Faith's problem on R-projectivity is independent of ZFC

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Overview

1. The role of projectivity and injectivity in representation theory
2. Baer Criterion for injectivity, and Faith's Problem on its dual
3. Shelah's Uniformization and the vanishing of Ext
4. The algebra of eventually constant sequences
5. Jensen's Diamond, and the independence of Faith's Problem of ZFC
6. Further examples in ZFC

Representable functors

Let R be a ring, $\mathrm{Mod}\text{–}R$ the category of all (right R-) modules, and $M \in \mathrm{Mod}\text{–}R$.

M induces two representable functors from $\mathrm{Mod}\text{–}R$ to $\mathrm{Mod}\text{–}\mathbb{Z}$: the covariant $F = \mathrm{Hom}_R(M, -)$, and the contravariant $G = \mathrm{Hom}_R(-, M)$.

Both these functors are left exact, i.e., given a short exact sequence

$$0 \to A \stackrel{\nu}{\to} B \stackrel{\pi}{\to} C \to 0$$

in $\mathrm{Mod}\text{–}R$, the sequences

$$0 \to F(A) \stackrel{F(\nu)}{\to} F(B) \stackrel{F(\pi)}{\to} F(C)$$

$$0 \to G(C) \stackrel{G(\pi)}{\to} G(B) \stackrel{G(\nu)}{\to} G(A)$$

are exact in $\mathrm{Mod}\text{–}\mathbb{Z}$.

Projective modules

Definition

M is a projective module, if $\mathrm{Hom}_R(M, -)$ is exact. Equivalently, for each short exact sequence of modules $0 \to A \to B \xrightarrow{\pi} C \to 0$ and each $f \in \mathrm{Hom}_R(M, C)$, there is a factorization of f through π:

$$\begin{tikzcd} & & M \arrow[d, dashed] \arrow[dr, "f"] & & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"] & C \arrow[r] & 0 \end{tikzcd}$$

The role of projective modules

- Free modules are projective, hence each module M can be presented as a homomorphic image of a projective module P: $0 \to K = \mathrm{Ker}(\pi) \to P \xrightarrow{\pi} M \to 0$.
- Iterating the presentation, we obtain a projective resolution of M: $\dots \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$.
- Given $N \in \mathrm{Mod}\text{–}R$, we can apply $\mathrm{Hom}_R(-, N)$ to the resolution above. The cohomology groups of the resulting complex are denoted by $\mathrm{Ext}^n_R(M, N)$ $(n \geq 0)$.
- These groups fit in a long exact sequence measuring the non-exactness of Hom: for $0 \to A \to B \to C \to 0$ a short exact sequence, we obtain the long one:

$$0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \to \mathrm{Ext}^1_R(C, N) \to$$

$$\to \mathrm{Ext}^1_R(B, N) \to \mathrm{Ext}^1_R(A, N) \to \mathrm{Ext}^2_R(C, N) \to \dots$$

Ext and extensions

- Let M be a module. Then M is projective, iff $\mathrm{Ext}^1_R(M,N) = 0$ for all $N \in \mathrm{Mod}\text{–}R$. Given a presentation $0 \to A \xrightarrow{\nu} B \to M \to 0$ of the module M with B projective, and a module N, we can employ the long exact sequence above and compute Ext by the formula $\mathrm{Ext}^1_R(M,N) \cong \mathrm{Hom}_R(A,N)/\mathrm{Im}(\mathrm{Hom}_R(\nu,N))$.

- $\mathrm{Ext}^1_R(M,N)$ can equivalently be defined as the group of equivalence classes of extensions of N by M, i.e., the short exact sequences $0 \to N \to X \to M \to 0$, with the equivalence is defined by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

  Addition is given by the Baer sum, and 0 is the equivalence class of the split extension $0 \to N \to N \oplus M \to M \to 0$.

The dual approach via injective modules

Definition

N is an injective module, if $\mathrm{Hom}_R(-, N)$ is exact. Equivalently, for each short exact sequence of modules $0 \to A \xrightarrow{\nu} B \to C \to 0$ and each $f \in \mathrm{Hom}_R(A, N)$, there is a factorization of f through ν:

$$\begin{tikzcd} & & N & & \\ 0 \arrow[r] & A \arrow[r, "\nu"] \arrow[ur, "f"] & B \arrow[u, dashed] \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

The role of injective modules

- Each module N is a submodule of an injective module I. Even in a 'minimal way', so N has an injective envelope $E(M)$.
- By iteration, we obtain a (minimal) injective coresolution of N: $0 \to N \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to \dots$.
- Given $M \in \mathrm{Mod}\text{–}R$, we can apply $\mathrm{Hom}_R(M, -)$ to the coresolution above. The cohomology groups of the resulting complex give an alternative way of defining $\mathrm{Ext}^n_R(M, N)$ ($n \geq 0$).
- N is injective, iff $\mathrm{Ext}^1_R(M, N) = 0$ for all $M \in \mathrm{Mod}\text{–}R$. This can be used to compute Ext via Hom using an injective copresentation of N.

The Baer Criterion for Injectivity

[Baer 1940]

The injectivity of a module M is equivalent to its R-injectivity, for any ring R and any module $M \in \mathrm{Mod}\text{–}R$.

Definition

M is *R-injective*, if for each right ideal I, all $f \in \mathrm{Hom}_R(I, M)$ extend to R:

$$\begin{tikzcd} & & M & & \\ 0 \arrow[r] & I \arrow[ur, "f"] \arrow[r, "\subseteq"] & R \arrow[u, dashed] \arrow[r] & R/I \arrow[r] & 0 \end{tikzcd}$$

Corollaries for the stucture theory

Definition

Let R be an integral domain. A module M is divisible, if $M.r = M$ for each $0 \neq r \in R$.
Equivalently, $\mathrm{Ext}^1_R(R/rR, M) = 0$ for each $0 \neq r \in R$.

Corollaries of Baer's Criterion

- injectivity = divisibility for R a Dedekind domain.
- Let R be a right noetherian ring. Then each injective module is uniquely a direct sum of modules isomorphic to $E(R/I)$ for some ideals I of R such that R/I uniform.
- (Matlis) Let R be a commutative noetherian ring. Then each injective module is uniquely a direct sum of modules isomorphic to $E(R/p)$ for some prime ideals p of R.

Faith's Problem

Original formulation

Algebra II - Ring Theory, Springer GMW 191, 1976.
Notes for Chapter 22 on p.175:

Sandomierski [64] showed that over a perfect ring R, that R is a "test module" for projectivity in a sense dual to the requirement for injectivity of a module M that maps of submodules of R into M can be lifted to maps of $R \to M$ (Baer's Criterion for Injectivity 3.41 (I, p. 157)). The characterization of all such rings is still an open problem.

Faith's problem in short

For what rings R does the Dual Baer Criterion hold, i.e.,
when is projectivity equivalent to R-projectivity?

Notation

Definition

M is R-projective, if for each right ideal I, all $f \in \mathrm{Hom}_R(M, R/I)$ factorize through π_I:

$$\begin{tikzcd} & & M \arrow[d, dashed] \arrow[dr, "f"] & & \\ 0 \arrow[r] & I \arrow[r, "\subseteq"] & R \arrow[r, "\pi_I"] & R/I \arrow[r] & 0 \end{tikzcd}$$

Equivalently, $\mathrm{Hom}_R(M, \pi_I)$ is surjective for each right ideal I of R.

Definition

The rings R such that projectivity of a module $M \in \mathrm{Mod}\text{–}R$ is equivalent to its R-projectivity are called right testing.

Projectivity relative to a module

Definition

Let M and B be modules. Then M is projective relative to B, or B-projective, if for each short exact sequence $0 \to A \to B \xrightarrow{\pi} C \to 0$, all $f \in \mathrm{Hom}_R(M, C)$ factorize through π:

$$\begin{tikzcd}
 & & M \arrow[d, dashed] \arrow[dr, "f"] & & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"] & C \arrow[r] & 0
\end{tikzcd}$$

Relative projectivity and finite direct sums

Lemma

Assume that M is B_i-projective for each $i < n$. Then M is B-projective, where $B = \bigoplus_{i<n} B_i$.

Proof: By induction on n.

For the inductive step, it suffices to consider the case when $B = B_0 \oplus B_1$.

We use the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_0 \cap K \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & K \arrow[r] \arrow[d, "\subseteq"'] & \rho(K) \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & B_0 \arrow[r, "\oplus"] \arrow[d, "\pi_{B_0 \cap K}"'] & B_0 \oplus B_1 \arrow[r, "\rho"] \arrow[d, "\pi_K"'] & B_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_0 + K/K \arrow[r, "\subseteq"] \arrow[d] & B_0 + B_1/K \arrow[r] \arrow[d] & B_0 + B_1/B_0 + K \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

R-projectivity for finitely generated modules

Lemma

Assume $M \in \mathrm{Mod}\text{–}R$ is finitely generated. Then M is R-projective, iff M is projective.

Proof: By the above, R-projectivity implies R^n-projectivity for each $n < \omega$. Assume M is n-generated. Then the identity map $1_M : M \to M$ factorizes through π in the free presentation of M:

$$\begin{tikzcd} & & M \arrow[d, dashed] \arrow[dr, "1_M"] & & \\ 0 \arrow[r] & K \arrow[r] & R^n \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

i.e., the free presentation splits. $\square$

R-projectivity of divisible modules

Lemma

Let R be an integral domain and M be a divisible module. Then M is R-projective.

Proof: Assume M is divisible and let I be a non-zero ideal of R such that $0 \neq \mathrm{Hom}_R(M, R/I)$. Then R/I contains a non-zero divisible submodule of the form J/I for an ideal $I \subsetneq J \subseteq R$. Let $0 \neq r \in I$. The r-divisibility of J/I yields $Jr + I = J$, but $Jr \subseteq I$, a contradiction. So $\mathrm{Hom}_R(M, R/I) = 0$ for each non-zero ideal I of R, and M is R-projective. $\square$

Corollary

$\mathbb{Q}$ is a countable $\mathbb{Z}$-projective, but not projective, $\mathbb{Z}$-module.

Perfect versus non-perfect rings

Definition

A ring R is right perfect, if R contains no infinite strictly decreasing chain of principal left ideals. E.g., each right artinian ring is right perfect.

The positive perfect case [Sandomierski 1964]

Each right perfect ring is right testing.

Some negative non-perfect cases

- [Hamsher 1966] If R is commutative and noetherian, then R is testing, iff R is artinian.
- If R is an integral domain, then R is testing, iff R is a field.
- [Puninski et. al. 2017] Let R be a semilocal right noetherian ring. Then R is right testing, iff R is right artinian.

Ladders and stationary sets

Ladders

Let κ be an uncountable cardinal of cofinality ω and $E \subseteq E_\omega$, where $E_\omega = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$.
A sequence $(n_\alpha \mid \alpha \in E)$ is a ladder system, if for each $\alpha \in E$, n_α is a ladder, i.e., a strictly increasing countable sequence $(n_\alpha(i) \mid i < \omega)$ consisting of non-limit ordinals such that $\sup_{i<\omega} n_\alpha(i) = \alpha$.

Stationary sets

Let κ be a regular uncountable cardinal.

- A subset $C \subseteq \kappa$ is called a club provided that C is closed in κ (i.e., $\sup(D) \in C$ for each subset $D \subseteq C$ such that $\sup(D) < \kappa$) and C is unbounded (i.e., $\sup(C) = \kappa$).
- $E \subseteq \kappa$ is stationary provided that $E \cap C \neq \emptyset$ for each club $C \subseteq \kappa$.

Example: E_ω is stationary in κ^+.

Shelah's Uniformization Principle (UP)

Uniformization of colorings

(UP_κ) There exist a stationary set $E \subseteq E_\omega$ and a ladder system $(n_\alpha \mid \alpha \in E)$, such that for each cardinal $\lambda < \kappa$ and each sequence $(h_\alpha \mid \alpha \in E)$ of maps (local λ-colorings) from ω to λ there exists a map (global λ-coloring) $f : \kappa^+ \to \lambda$, such that for each $\alpha \in E$, $f(n_\alpha(i)) = h_\alpha(i)$ for almost all $i < \omega$.

(UP) UP_κ holds for each uncountable cardinal κ of cofinality ω.

Theorem (Eklof-Shelah 1991)

UP is consistent with ZFC + GCH.

Faith's problem under Shelah's uniformization

[T. 1996]

Let R be a non-right perfect ring and κ an uncountable cardinal of cofinality ω, such that $\mathrm{card}(R) < \kappa$ and UP_κ holds. Then there exists a κ^+-generated module M_κ of projective dimension 1 such that $\mathrm{Ext}^1_R(M_\kappa, I) = 0$ for each right ideal I of R.

[Puninski et al. 2017]

The module M_κ is R-projective, but not projective.

Proof: $\mathrm{Hom}_R(M_\kappa, R) \overset{\mathrm{Hom}_R(M_\kappa,\pi_I)}{\longrightarrow} \mathrm{Hom}_R(M_\kappa, R/I) \to \mathrm{Ext}^1_R(M_\kappa, I) = 0$ is an exact sequence. So $\mathrm{Hom}_R(M_\kappa, \pi_I)$ is surjective for each right ideal I of R, and M_κ is R-projective. □

Corollary

Assume UP. Then right testing rings coincide with the right perfect ones.

The construction of the module M_κ

M_κ is defined by a free presentation

$$(*) \qquad 0 \to G \xrightarrow{\nu} F \to M_\kappa \to 0,$$

where $F = \bigoplus_{\alpha<\kappa^+} F_\alpha$, $F_\alpha = R^{(\omega)}$ for $\alpha \in E$, and $F_\alpha = R$ otherwise.

Let 1_α be the canonical free generator of F_α for $\alpha \notin E$, and $\{1_{\alpha,i} \mid i < \omega\}$ the canonical free basis of F_α for $\alpha \in E$.

Let $R \supsetneq Ra_0 \supsetneq Ra_1a_0 \supsetneq \dots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_0 \supsetneq \dots$ be a strictly decreasing chain of principal left ideals of R.

For $\alpha \in E$ and $i < \omega$, we define $g_{\alpha,i} = 1_{\nu_{\alpha(i)}} - 1_{\alpha,i} + 1_{\alpha,i+1}.a_i$, and $G = \bigoplus_{\alpha \in E, i<\omega} g_{\alpha,i} R$.

Lemma

The presentation $(*)$ above is free, but non-split, whence the projective dimension of $M_\kappa = F/G$ equals 1.

The vanishing of $\mathrm{Ext}^1_R(M_\kappa, I)$

Recall that $\mathrm{Ext}^1_R(M, I) = 0$, iff $\mathrm{Hom}_R(G, I) = \mathrm{Im}(\mathrm{Hom}_R(\nu, I))$, iff each homomorphism $\varphi \in \mathrm{Hom}_R(G, I)$ extends to some $\psi \in \mathrm{Hom}_R(F, I)$.

Let $\lambda = \mathrm{card}(I)$. Then $\lambda < \kappa$, and h defines a local λ-coloring from ω to λ by $h_\alpha(i) = \varphi(g_{\alpha,i})$.

The global λ-coloring $f : \kappa^+ \to \lambda$ provided by (UP_κ) can be used to define $\psi \in \mathrm{Hom}_R(F, I)$ so that $\varphi = \psi \restriction G$, i.e., prove that $\mathrm{Ext}^1_R(M_\kappa, I) = 0$. $\square$

Remark: The global coloring f coincides with each of the local colorings h_α **almost everywhere**, *while we need ψ to restrict to φ* **everywhere**. *This can be fixed using the extra space provided by F_α (recall that for $\alpha \in E$, F_α has rank $\aleph_0$ rather than 1).*

Jensen's functions

Let κ be a regular uncountable cardinal.

- Let A be a set of cardinality $\leq \kappa$. An increasing continuous chain, $\mathcal{A} = (A_\alpha \mid \alpha < \kappa)$, consisting of subsets of A of cardinality $< \kappa$, such that $A_0 = 0$ and $A = \bigcup_{\alpha<\kappa} A_\alpha$, is called a κ-filtration of the set A.
- Let E be a stationary subset of κ. Let A and B be sets of cardinality $\leq \kappa$. Let $\mathcal{A}$ and $\mathcal{B}$ be κ-filtrations of A and B, respectively. For each $\alpha < \kappa$, let $c_\alpha : A_\alpha \to B_\alpha$ be a map. Then $(c_\alpha \mid \alpha < \kappa)$ are Jensen-functions provided that for each map $c : A \to B$, the set $E(c) = \{\alpha \in E \mid c \restriction A_\alpha = c_\alpha\}$ is stationary in κ.

Theorem (Jensen 1972)

Assume Gödel's Axiom of Constructibility ($V = L$). Let κ be a regular uncountable cardinal, $E \subseteq \kappa$ a stationary subset of κ, and A and B sets of cardinality $\leq \kappa$. Let $\mathcal{A}$ and $\mathcal{B}$ be κ-filtrations of A and B, respectively. Then there exist Jensen-functions $(c_\alpha \mid \alpha < \kappa)$.

The algebra of eventually constant sequences

Let K be a field. Denote by $\mathcal{E}(K)$ the unital K-subalgebra of K^ω generated by $K^{(\omega)}$. In other words, $\mathcal{E}(K)$ is the subalgebra of K^ω consisting of all eventually constant sequences in K^ω.

Basic properties

Let $R = \mathcal{E}(K)$.

- R is a commutative von Neumann regular hereditary semiartinian ring of Loewy length 2 with $\mathrm{Soc}(R) = K^{(\omega)}$.
- R is not perfect.
- A module M is R-projective, if each $f \in \mathrm{Hom}_R(M, \mathrm{Soc}(R))$ factors through the canonical projection $\pi : R \to R/\mathrm{Soc}(R)$.
- If $M \in \mathrm{Mod}\text{–}R$ is countably generated, then M is R-projective, iff M is projective.

Faith's problem under $\mathbf{V = L}$

Theorem (T. 2017)

Assume $V = L$. Let K be a field of cardinality $\leq 2^{\omega}$, and $R = \mathcal{E}(K)$. Then R is right testing.

Sketch of proof

Let M be an R-projective module and κ be the minimal number of R-generators of M. The proof is by induction on κ:

If $\kappa \leq \aleph_0$, then we use the last basic property above.

If κ is regular and uncountable, then M can be expressed as the union of a continuous chain of its $< \kappa$-generated submodules $\mathcal{M} = (M_\alpha \mid \alpha < \kappa)$. W.l.o.g., we can assume that if M_β / M_α is not R-projective, then $M_{\alpha+1}/M_\alpha$ is not R-projective, too. Using Jensen-functions, one proves that the set $E = \{\alpha < \kappa \mid M_{\alpha+1}/M_\alpha$ is not R-projective $\}$ is not stationary in κ. Then we can select a continuous subchain $\mathcal{M}'$ of $\mathcal{M}$ such that $M'_{\alpha+1}/M'_\alpha$ is R-projective for each $\alpha < \kappa$. By the inductive premise, $M'_{\alpha+1}/M'_\alpha$ is projective, and hence $M'_{\alpha+1} = M'_\alpha \oplus P_\alpha$ for a $< \kappa$-generated projective module P_α. Then $M = M'_0 \oplus \bigoplus_{\alpha<\kappa} P_\alpha$ is projective.

If κ is singular, we use a version of Shelah's Compactness Theorem pro projective modules. $\square$

Faith's problem is independent of ZFC + GCH

The statement 'There exists a right testing, but non-right perfect ring' is independent of ZFC + GCH.

Proof: Assuming UP, we get that each right testing ring is right perfect, but V = L implies that the non-right perfect ring of all eventually constant sequences $\mathcal{E}(K)$ is right testing. □

Further examples

Example 1

Let R be an infinite direct product of skew-fields. Then all R-projective modules are non-singular, and the Dual Baer Criterion holds for all countably generated modules.

Example 2

Let R be a von Neumann regular right self-injective ring which is purely infinite (e.g., R is the endomorphism ring of any infinite dimensional right vector space over a skew-field). Then the Dual Baer Criterion holds for all $\leq 2^{\aleph_0}$-presented modules of projective dimension ≤ 1.

Chronology of references

F.Sandomierski, *Relative Injectivity and Projectivity*, Penn State U. 1964.

C.Faith, *Algebra II. Ring Theory*, GMW 191, Springer-Verlag, Berlin 1976.

P.C.Eklof, S.Shelah, *On Whitehead modules*, J.Algebra 142(1991), 492-510.

J.Trlifaj, *Whitehead test modules*, TAMS 348(1996), 1521-1554.

H.Q.Dinh, C.J.Holston, D.V.Huynh, *Quasi-projective modules over prime hereditary noetherian V-rings are projective or injective*, J.Algebra 360(2012), 87-91.

H.Alhilali, Y.Ibrahim, G.Puninski, M.Yousif, *When R is a testing module for projectivity?*, J.Algebra 484(2017), 198-206.

J.Trlifaj, *Faith's problem on R-projectivity is undecidable*, arXiv:1710.10465v1.

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