Some representation theory arising from set-theoretic homological algebra

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Maurice Auslander International Conference

Woods Hole, April 27th, 2016

I. Decomposition, deconstruction, and their limitations

Classic structure theory: direct sum decompositions

A class of modules $\mathcal{C}$ is decomposable, provided that there is a cardinal κ such that each module in $\mathcal{C}$ is a direct sum of strongly $< \kappa$-presented modules from $\mathcal{C}$.

[Kaplansky]

1. The class $\mathcal{P}_0$ of all projective modules is decomposable.

[Faith-Walker]

2. The class $\mathcal{I}_0$ of all injective modules is decomposable iff R is a right noetherian ring.

[Huisgen-Zimmermann]

3. Mod-R is decomposable iff R is a right pure-semisimple ring.
In fact, if M is a module such that $\mathrm{Prod}(M)$ is decomposable, then M is Σ-pure-injective.

Such examples, however, are rare in general – most classes of modules are not decomposable.

Example

Assume that the ring R is not right perfect, that is, there is a strictly decreasing chain of principal left ideals

$$Ra_0 \supsetneq \cdots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_o \supsetneq \dots$$

Then the class $\mathcal{F}_0$ of all flat modules is not decomposable.

Example

There exist arbitrarily large indecomposable flat abelian groups.

Transfinite extensions

Let $\mathcal{A} \subseteq \mathrm{Mod}{-}R$. A module M is $\mathcal{A}$-filtered (or a transfinite extension of the modules in $\mathcal{A}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$.

Notation: $M \in \mathrm{Filt}(\mathcal{A})$. A class $\mathcal{A}$ is filtration closed if $\mathrm{Filt}(\mathcal{A}) = \mathcal{A}$.

Eklof Lemma

${}^\perp\mathcal{C} = \mathrm{KerExt}^1_R(-,\mathcal{C})$ is filtration closed for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

Deconstructible classes

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{<\kappa})$ where $\mathcal{A}^{<\kappa}$ denotes the class of all strongly $< \kappa$-presented modules from $\mathcal{A}$.

All decomposable classes are deconstructible.

For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.

[Eklof-T.]

More in general, for each set of modules $\mathcal{S}$, the class ${}^\perp(\mathcal{S}^\perp)$ is deconstructible. Here, $\mathcal{S}^\perp = \mathrm{KerExt}^1_R(\mathcal{S}, -)$.

Approximations for relative homological algebra

A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through f:

$$\begin{tikzcd} A \arrow[r, "f"] & M \\ A' \arrow[u, dashed] \arrow[ur, "f'"'] & \end{tikzcd}$$

The map f is an $\mathcal{A}$–precover of M.

If f is moreover right minimal (that is, f factorizes through itself only by an automorphism of A), then f is an $\mathcal{A}$–cover of M.

If $\mathcal{A}$ provides for covers for all modules, then $\mathcal{A}$ is called a covering class.

The abundance of approximations

[Enochs], [Šťovíček]

- All deconstructible classes are precovering.
- All precovering classes closed under direct limits are covering.

In particular, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is precovering for any set of modules $\mathcal{S}$.
Note: If $R \in \mathcal{S}$, then ${}^{\perp}(\mathcal{S}^{\perp})$ coincides with the class of all direct summands of $\mathcal{S}$-filtered modules.

Flat cover conjecture

$\mathcal{F}_0$ is covering for any ring R, and so are the classes $\mathcal{F}_n$ for each $n > 0$.

The classes $\mathcal{P}_n$ $(n \geq 0)$ are precovering. ...

Bass modules

Let R be a ring and $\mathcal{C}$ be a class of countably presented modules. $\varinjlim_\omega \mathcal{C}$ denotes the class of all Bass modules over $\mathcal{C}$, that is, the modules B that are countable direct limits of modules from $\mathcal{C}$.
W.l.o.g., such B is the direct limit of a chain

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} F_i \xrightarrow{f_i} F_{i+1} \xrightarrow{f_{i+1}} \dots$$

with $F_i \in \mathcal{C}$ and $f_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

The classic Bass module

Let $\mathcal{C}$ be the class of all countably generated projective modules. Then the Bass modules coincide with the countably presented flat modules.
If R is not right perfect, then a classic instance of such a Bass module B arises when $F_i = R$ and f_i is the left multiplication by a_i ($i < \omega$) where $Ra_0 \supsetneq \dots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1} a_n \dots a_o \supsetneq \dots$ is strictly decreasing.

Flat Mittag-Leffler modules

[Raynaud-Gruson]

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

The class of all flat Mittag-Lefler modules is denoted by $\mathcal{FM}$.

$\mathcal{P}_0 \subseteq \mathcal{FM} \subseteq \mathcal{F}_0$.
$\mathcal{FM}$ is filtration closed, and it is closed under pure submodules.

[Raynaud-Gruson]

$M \in \mathcal{FM}$, iff each countable subset of M is contained in a countably generated projective and pure submodule of M.
In particular, all countably generated modules in $\mathcal{FM}$ are projective.

Flat Mittag-Leffler modules and approximations

Theorem (Angeleri-Šaroch-T.)

Assume that R is not right perfect. Then the class $\mathcal{FM}$ is not precovering, and hence not deconstructible.

Idea of proof: Choose a non-projective Bass module B over $\mathcal{P}_0^{<\omega}$, and prove that B has no $\mathcal{FM}$-precover.

The main tool: Tree modules.

II. Tree modules and their applications

The trees

Let κ be an infinite cardinal, and T_κ be the set of all finite sequences of ordinals $< \kappa$, so

$$T_\kappa = \{\tau : n \to \kappa \mid n < \omega\}.$$

Partially ordered by inclusion, T_κ is a tree, called the tree on κ.

Let $\mathrm{Br}(T_\kappa)$ denote the set of all branches of T_κ. Each $\nu \in \mathrm{Br}(T_\kappa)$ can be identified with an ω-sequence of ordinals $< \kappa$:

$$\mathrm{Br}(T_\kappa) = \{\nu : \omega \to \kappa\}.$$

$|T_\kappa| = \kappa$ and $|\mathrm{Br}(T_\kappa)| = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of τ for each $\tau \in T_\kappa$.

Decorating trees by Bass modules

Let $D := \bigoplus_{\tau \in T_\kappa} F_{\ell(\tau)}$, and $P := \prod_{\tau \in T_\kappa} F_{\ell(\tau)}$.

For $\nu \in \mathrm{Br}(T_\kappa)$, $i < \omega$, and $x \in F_i$, we define $x_{\nu i} \in P$ by

$$\pi_{\nu \restriction i}(x_{\nu i}) = x,$$

$$\pi_{\nu \restriction j}(x_{\nu i}) = g_{j-1} \dots g_i(x) \text{ for each } i < j < \omega,$$

$$\pi_\tau(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_\tau \in \mathrm{Hom}_R(P, F_{\ell(\tau)})$ denotes the τth projection for each $\tau \in T_\kappa$.

Let $X_{\nu i} := \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of P isomorphic to F_i.

The tree modules

Let $X_\nu := \sum_{i<\omega} X_{\nu i}$, and $G := \sum_{\nu \in \mathrm{Br}(T_\kappa)} X_\nu$.

Basic properties

- $D \subseteq G \subseteq P$.
- There is a 'tree module' exact sequence

$$0 \to D \to G \to B^{(\mathrm{Br}(T_\kappa))} \to 0.$$

- G is a flat Mittag-Leffler module.

Proof of the Theorem

Assume there exists a $\mathcal{FM}$-precover $f : F \to B$ of the classic Bass module B. Let $K = \mathrm{Ker}(f)$, so we have an exact sequence

$$0 \to K \hookrightarrow F \xrightarrow{f} B \to 0.$$

Let κ be an infinite cardinal such that $|R| \leq \kappa$ and $|K| \leq 2^\kappa = \kappa^\omega$.

Consider the 'tree module' exact sequence

$$0 \to D \hookrightarrow G \to B^{(2^\kappa)} \to 0,$$

so $G \in \mathcal{FM}$ and D is a free module of rank κ. Clearly, $G \in \mathcal{P}_1$.

Let $\eta : K \to E$ be a $\{G\}^\perp$-preenvelope of K with a $\{G\}$-filtered cokernel.

Consider the pushout

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r, "\subseteq"] \arrow[d, "\eta"'] & F \arrow[r, "f"] \arrow[d, "\varepsilon"'] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "\subseteq"] \arrow[d] & P \arrow[r, "g"] \arrow[d] & B \arrow[r] & 0 \\
 & \mathrm{Coker}(\eta) \arrow[r, "\cong"] \arrow[d] & \mathrm{Coker}(\varepsilon) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $P \in \mathcal{FM}$. Since f is an $\mathcal{FM}$-precover, there exists $h : P \to F$ such that $fh = g$. Then $f = g\varepsilon = fh\varepsilon$, whence $K + \mathrm{Im}(h) = F$. Let $h' = h \restriction E$. Then $h' : E \to K$ and $\mathrm{Im}(h') = K \cap \mathrm{Im}(h)$.

Consider the restricted exact sequence

$$0 \longrightarrow \operatorname{Im}(h') \xrightarrow{\subseteq} \operatorname{Im}(h) \xrightarrow{f\restriction \operatorname{Im}(h)} B \longrightarrow 0.$$

As $E \in G^{\perp}$ and $G \in \mathcal{P}_1$, also $\operatorname{Im}(h') \in G^{\perp}$.

Applying $\operatorname{Hom}_R(-, \operatorname{Im}(h'))$ to the 'tree-module' exact sequence above, we obtain the exact sequence

$$\operatorname{Hom}_R(D, \operatorname{Im}(h')) \to \operatorname{Ext}^1_R(B, \operatorname{Im}(h'))^{2^\kappa} \to 0$$

where the first term has cardinality only $\leq |K|^\kappa \leq 2^\kappa$, so the second term must be zero.

This yields $\operatorname{Im}(h') \in B^{\perp}$. Then $f \restriction \operatorname{Im}(h)$ splits, and so does the $\mathcal{FM}$-precover f, a contradiction with $B \notin \mathcal{FM}$. □

The role of the Bass modules

Lemma (Šaroch)

Let $\mathcal{C}$ be a class of countably presented modules, and $\mathcal{L}$ the class of all 'locally $\mathcal{C}$-free' modules.

Let B be a Bass module over $\mathcal{C}$ such that B is not a direct summand in a module from $\mathcal{L}$.

Then B has no $\mathcal{L}$-precover.

III. A generalization via tilting theory

Large tilting modules

T is a (large) tilting module provided that

- T has finite projective dimension,
- $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for each cardinal κ, and
- there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \mathrm{Add}(T)$, i.e., T_i is a direct summand of a (possibly infinite) direct sum of copies of T .

$\mathcal{B} = \{T\}^{\perp_\infty} = \bigcap_{1<i} \mathrm{KerExt}^i_R(T, -)$ the right tilting class of T.
$\mathcal{A} = {}^{\perp}\mathcal{B}$ the left tilting class of T.

- $\mathcal{A} \cap \mathcal{B} = \mathrm{Add}(T)$.
- Right tilting classes coincide with the classes of finite type, that is, they have the form $\mathcal{S}^\perp$ where $\mathcal{S}$ is a set of strongly finitely presented modules of bounded projective dimension.
- $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{\leq\omega})$, hence $\mathcal{A}$ is precovering. Moreover, $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$.

Σ-pure split tilting modules

A module M is Σ-pure split provided that each pure embedding $N' \hookrightarrow N$ with $N \in \mathrm{Add}(M)$ splits.

[Angeleri-T.]

A tilting module T is Σ-pure split, iff $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$, iff $\mathcal{A}$ closed under direct limits.

Examples

Let $T = R$. Then T is a tilting module of projective dimension 0, and T is Σ-pure split iff R is a right perfect ring.

Each Σ-pure injective tilting module is Σ-pure split.
Each finitely generated tilting module over any artin algebra is Σ-pure injective.

Locally T-free modules

Let R be a ring and T a tilting module.

A module M is locally T-free provided that M possesses a set $\mathcal{H}$ of submodules such that

- $\mathcal{H} \subseteq \mathcal{A}^{\leq \omega}$,
- each countable subset of M is contained in an element of $\mathcal{H}$,
- $\mathcal{H}$ is closed under unions of countable chains.

Let $\mathcal{L}$ denote the class of all locally T-free modules.

Note: If M is countably generated, then M is locally T-free, iff $M \in \mathcal{A}^{\leq \omega}$.

Flat-Mittag Leffler modules revisited

For any ring R and any tilting module T, we have

$$\mathcal{A} \subseteq \mathcal{L} \subseteq \varinjlim \mathcal{A}^{<\omega}.$$

The 0-dimensional case

Let R be an arbitrary ring and $T = R$. Then

$$\mathcal{A} = \mathcal{P}_0 \subseteq \mathcal{L} = \mathcal{FM} \subseteq \varinjlim \mathcal{A}^{<\omega} = \mathcal{F}_0.$$

Locally T-free modules and approximations

Theorem

Let R be a ring and T be a tilting module. Then TFAE:

1. *$\mathcal{L}$ is (pre)covering.*
2. *$\mathcal{L}$ is deconstructible.*
3. *T is Σ-pure split.*

Note: The theorem on flat Mittag-Leffler modules stated earlier is just the particular case of $T = R$.

The role of Bass modules, and Enochs' Conjecture

Theorem

$\mathcal{L}$ is (pre)covering, iff $\mathcal{A}$ is closed under direct limits, iff $B \in \mathcal{A}$ for each Bass module B over $\mathcal{A}^{<\omega}$ (i.e., $\varinjlim_{\omega}(\mathcal{A}^{<\omega}) \subseteq \mathcal{A}$).

Enochs' Conjecture

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is covering, iff $\mathcal{C}$ is precovering and closed under direct limits.

Corollary

The Enochs' Conjecture holds for all left tilting classes of modules.

A finite dimensional example

Let R be an indecomposable hereditary finite dimensional algebra of infinite representation type.
Then there is a partition of ind-R into three sets:

q ... the indecomposable preinjective modules

p ... the indecomposable preprojective modules

t ... the regular modules (the rest).

Then $p^\perp$ is a right tilting class (and $M \in p^\perp$, iff M has no non-zero direct summands from p).

The tilting module T inducing $p^\perp$ is called the Lukas tilting module.
The left tilting class of T is the class of all Baer modules.
The locally T-free modules are called locally Baer modules.

Non-precovering classes of locally Baer modules

Theorem

- *The class of all Baer modules coincides with Filt(p).*
- *The Lukas tilting module T is countably generated, but has no finite dimensional direct summands, and it is not Σ-pure split.*
  So the class $\mathcal{L}$ is not precovering (and hence not deconstructible).

The Bass modules behind the scene

The relevant Bass modules can be obtained as unions of the chains

$$P_0 \overset{f_0}{\hookrightarrow} P_1 \overset{f_1}{\hookrightarrow} \dots \overset{f_{i-1}}{\hookrightarrow} P_i \overset{f_i}{\hookrightarrow} P_{i+1} \overset{f_{i+1}}{\hookrightarrow} \dots$$

such that all the P_i are preprojective (i.e., in add(p)), but the cokernels of all the f_i are regular (i.e., in add(t)).

IV. Tree modules and the Auslander problem

Almost split maps and sequences

Definition

Let R be a ring and N be a module. A morphism of modules $f : M \longrightarrow N$ is right almost split, provided that the following are equivalent for each morphism $g : P \longrightarrow N$:

- g factorizes through f,
- g is not a split epimorphism.

Dually, left almost split morphisms $f' : N' \longrightarrow M'$ are defined.

A short exact sequence of modules $0 \longrightarrow N' \xrightarrow{f'} M \xrightarrow{f} N \longrightarrow 0$ is almost split, if f and f' are right and left almost split morphisms, respectively.

Theorem (Auslander)

Let N be an (indecomposable) finitely presented module with local endomorphism ring. Then there exists a right almost split morphism $f : M \longrightarrow N$. If N is not projective, then there even exists an almost split sequence as above.

Auslander's problem and generalized tree modules

Auslander'1975, in Proc. 2nd Conf. Univ. Oklahoma

Are there further examples of right almost split morphisms in Mod-R?

A negative answer has recently been given using (generalized) tree modules:

Theorem (Šaroch'2015)

Let R be a ring and N be a module. TFAE:

1. *There exists a right almost split morphism $f : M \to N$.*
2. *N is finitely presented, and its endomorphism ring is local.*

Corollary

Let R be a ring and $0 \to P \to M \to N \to 0$ be an almost split sequence in Mod-R. Then N is finitely presented with local endomorphism ring, and P is pure-injective.

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