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★**Preconditioning and the conjugate gradient method in the context of solving PDEs.**
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The book under review is devoted to presenting theoretical results and practical algorithms for the numerical solution of partial differential equations (PDEs). When solving such problems, there are many subtasks which come from very different areas of mathematics: theoretical mathematical disciplines (analysis of initial and boundary value PDE problems, functional analysis, and calculus of variations) concerned with qualitative questions of existence, uniqueness, and asymptotic behavior of solutions; then function spaces, approximation theory, quadrature and mesh generation for discretization of infinite-dimensional problems; and, finally, different tools from matrix computations for efficient and numerically stable implementations on highly parallel computers of finite-dimensional problems resulting from the discretization. The authors consider all these steps as inseparable for the entire solution process. A lot of papers, books and book chapters have been written about each of the above-mentioned items individually, or about a few in combination. But, as far as I know, nothing has been written about all of them together, with a unified vision. For this reason alone, the authors, coming from the school on numerical mathematics of the Czech Republic, should be thanked and congratulated. Also, the monograph (or at least, some chapters) can be used as a textbook for master's degree students (or even undergraduates) for a course on numerical solutions of PDEs.

This monograph is structured into thirteen short chapters.

Chapter 1 is devoted to presenting the problem under consideration, namely a subclass of linear elliptic second-order boundary-value problems (BVPs) analysed via the Lax-Milgram lemma, i.e. solving the functional equation

$$\mathcal{A}u = d, \tag{1}$$

with $\mathcal{A}\colon \to V^\sharp$, $\mathcal{A}$ self-adjoint, $d \in V^\sharp$, V a real infinite-dimensional Hilbert space and $V^\sharp$ its dual, or as in numerical linear algebra, solving

$$Ax = b, \tag{2}$$

with A is an $n \times n$ symmetric and positive definite matrix and $b \in \mathbb{R}^n$.

The diagram where *Preconditioned CG* (Conjugate Gradient) is at the intersection of *PDE Analysis of the mathematical model*, *Discretization*, and *Iterative computation*, is found to be more relevant than the standard way of solving real-world problems: *Real-world problem* $\to$ *Mathematical model* $\to$ *Discretization* $\to$ *Computation*, and the other way around. This is the reason behind the authors' motivation for the current book.

Chapter 2 covers the basic concepts (especially various equivalent norms) regarding the Sobolev space $H_1(\Omega)$ and the problems that can be covered by the equation (1), and it emphasizes the differences between the qualitative analysis of PDEs and the quantitative numerical approximations of their solutions.

Chapter 3 is concerned with the elements of functional analysis for the setting of equation (1), recalling several basic results connected also with the well-posedness of (1): the Riesz representation theorem, the Lax-Milgram lemma, and the role of the symmetry of bilinear forms; this part can also be found in standard textbooks.

In Chapter 4 the authors specify the preconditioning for equation (1) when defined by a choice of an appropriate inner product. Chapter 5 presents the conjugate gradient method in Hilbert spaces with the analytic view linking the Vorobyev moment problem, CG, and Gauss-Christoffel quadrature. In Chapter 6, the authors address CG problems in the finite-dimensional Hilbert space setting, by writing the standard matrix formulation of the preconditioned CG. Chapter 7 is about Galerkin discretization, while Chapter 8 gives a general description of the fact that algebraic preconditioning of the linear algebraic system associated with the operator preconditioning is equivalent to orthonormalization of the discretization basis in the given finite-dimensional Hilbert space. In Chapter 9 the authors recall the consistency, stability, and convergence of the discretization schemes, while the next chapter deals with error evaluation in numerical PDEs. Here the authors point out that the spatial distribution of the discretization and algebraic error can be quite different. One of the reasons is discussed in Chapter 11. In the next chapter the authors discuss the need to take care of all sources of error when performing the a posteriori analysis. The last chapter is actually a summary of the book.

The book has a rich bibliography of 204 references (about 14 of them are the authors' papers) and concludes with a subject index.

The monograph is without any doubt a very carefully prepared one, and researchers interested in solving PDEs by means of preconditioning and conjugate gradient method will benefit from exposure to this book. *Elena Pelican*