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A Simple Formula for the Generalized Spectrum of Second Order Self-Adjoint Differential Operators*

Bjørn Fredrik Nielsen[†]
Zdeněk Strakoš[‡]

Abstract. We analyze the spectrum of the operator $\Delta^{-1}[\nabla \cdot (K\nabla u)]$ subject to homogeneous Dirichlet or Neumann boundary conditions, where Δ denotes the Laplacian and $K = K(x, y)$ is a symmetric tensor. Our main result shows that this spectrum can be derived from the spectral decomposition $K = Q\Lambda Q^T$, where $Q = Q(x, y)$ is an orthogonal matrix and $\Lambda = \Lambda(x, y)$ is a diagonal matrix. More precisely, provided that K is continuous, the spectrum equals the convex hull of the ranges of the diagonal function entries of Λ. The domain involved is assumed to be bounded and Lipschitz. In addition to studying operators defined on infinite-dimensional Sobolev spaces, we also report on recent results concerning their discretized finite-dimensional counterparts. More specifically, even though $\Delta^{-1}[\nabla\cdot(K\nabla u)]$ is not compact, it turns out that every point in the spectrum of this operator can, to an arbitrary accuracy, be approximated by eigenvalues of the associated generalized algebraic eigenvalue problems arising from discretizations. Our theoretical investigations are illuminated by numerical experiments. The results presented in this paper extend previous analyses which have addressed elliptic differential operators with scalar coefficient functions. Our investigation is motivated by both preconditioning issues (efficient numerical computations) and the need to further develop the spectral theory of second order PDEs (core analysis).

Key words. second order PDEs, generalized eigenvalues, spectrum, tensors, preconditioning

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1. Main Result.

For simple domains, the eigenfunctions and eigenvalues of the Laplacian Δ can be characterized in terms of trigonometric functions. Similar analytic information is not available for general second order differential operators $\nabla \cdot (K\nabla u)$. However, in [9, 20] the authors show that the spectrum of the preconditioned operator

$$\Delta^{-1}[\nabla \cdot (k\nabla u)] \quad \text{for } (x, y) \in \Omega, \tag{1.1}$$

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[†]Faculty of Science and Technology, Norwegian University of Life Sciences, NO-1432 Ås, Norway (bjorn.f.nielsen@nmbu.no).

[‡]Faculty of Mathematics and Physics, Charles University, 186 75 Prague 8, Czech Republic (strakos@karlin.mff.cuni.cz).

where k is a uniformly positive scalar function, can be analyzed in detail. More specifically, if k is continuous, then the range

$$k(\Omega) = \{k(x,y),\, (x,y) \in \Omega\}$$

of k is contained in the spectrum of the operator (1.1). Here, $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, $k = k(x,y)$, $u = u(x,y)$, and one employs homogeneous Dirichlet boundary conditions. Furthermore, for discretized problems, assuming that k is bounded and piecewise continuous, the function values of k over the patches defined by the discretization basis functions provide accurate approximations of the generalized eigenvalues.

This raises two questions. First, using homogeneous Dirichlet or Neumann boundary conditions, can *the whole spectrum* of the operator (1.1) be expressed in terms of a simple formula and can such a result be further extended? Second, what is the relationship between the spectrum of the preconditioned differential operator and the generalized eigenvalues of its discrete counterpart? The main purpose of this paper is to address these two questions.

Concerning the first question, we extend the results published in [9, 20] to second order differential operators which involve a symmetric tensor $K(x,y)$ instead of a scalar function $k(x,y)$: Consider a symmetric real valued tensor function $K : \Omega \to \mathbb{R}^{2\times 2}$ with bounded Lebesgue integrable functions entries and with the spectral decomposition

$$\begin{aligned} &K(x,y) = Q(x,y)\Lambda(x,y)Q^T(x,y), \quad (x,y) \in \Omega, \\ &\Lambda(x,y) = \begin{bmatrix} \kappa_1(x,y) & 0 \\ 0 & \kappa_2(x,y) \end{bmatrix}, \quad QQ^T = Q^TQ = I. \end{aligned} \tag{1.2}$$

Defining the operators $\mathcal{L}$, $\mathcal{A} : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ as

$$\langle \mathcal{L}\phi, \psi \rangle = \int_\Omega \nabla\phi \cdot \nabla\psi, \quad \phi, \psi \in H_0^1(\Omega), \tag{1.3}$$

$$\langle \mathcal{A}\phi, \psi \rangle = \int_\Omega K\nabla\phi \cdot \nabla\psi, \quad \phi, \psi \in H_0^1(\Omega), \tag{1.4}$$

we characterize the spectrum of the preconditioned operator

$$\mathcal{L}^{-1}\mathcal{A} : H_0^1(\Omega) \to H_0^1(\Omega), \tag{1.5}$$

defined as

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) \equiv \left\{\lambda \in \mathbb{C};\ \lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A} \text{ does not have a bounded inverse}\right\}. \tag{1.6}$$

More specifically, this paper proves the following result, which contains the formula referred to in its title.[1]

Theorem 1.1 (spectrum of the preconditioned operator). *Consider an open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. Assume that the tensor K is symmetric and continuous throughout the closure $\overline{\Omega}$. Then the spectrum of the operator $\mathcal{L}^{-1}\mathcal{A}$, defined in* (1.3)–(1.6), *equals*

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) = \mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})), \tag{1.7}$$

[1]Homogeneous Neumann boundary conditions are treated in section 6.

where

$$\mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})) = \left[\inf_{(x,y)\in\overline{\Omega}} \min_{i=1,2} \kappa_i(x,y),\ \sup_{(x,y)\in\overline{\Omega}} \max_{i=1,2} \kappa_i(x,y) \right]. \tag{1.8}$$

Note that this theorem extends the results in [20] in several ways. It holds for second order differential operators with definite, indefinite, and semidefinite tensors. Moreover, instead of the inclusion proved for the scalar case in [20], it shows that the spectrum actually equals the interval (1.8) determined by $K(x, y)$. This answers the first question posed above.

As for the second question, let $\mathbf{L_n}$ and $\mathbf{A_n}$ be the matrices that arise from discretizations of the operators (1.3) and (1.4), respectively, using an n-dimensional subspace of $H_0^1(\Omega)$. Then the numerical approximation of $\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A})$ is typically computed via the generalized algebraic eigenvalue problem

$$\mathbf{A_n v} = \lambda \mathbf{L_n v}, \quad \mathbf{v} \in \mathbb{R}^n. \tag{1.9}$$

In the numerical PDE literature, this generalized algebraic eigenvalue problem is often associated with the generalized PDE eigenvalue problem

$$\begin{aligned} \nabla \cdot (K\nabla u) &= \lambda \Delta u \quad \text{for } (x,y) \in \Omega, \\ u &= 0 \quad \text{for } (x,y) \in \partial\Omega. \end{aligned} \tag{1.10}$$

The relationship among (1.9), (1.10), and the numerical approximation of the whole spectrum (1.6) is, however, rather involved: The point spectrum of a noncompact operator represents only a part of its spectrum. This is, in particular, true for continuously invertible operators defined on infinite-dimensional Hilbert (Sobolev) spaces, which cannot be compact. Therefore, the generalized eigenvalue problem (1.10) does not determine the whole spectrum (1.6). On the other hand, the eigenvalues of the generalized eigenvalue problem (1.9) *do* asymptotically approximate the whole spectrum (1.6). We will discuss these issues in section 8.

2. Introduction to the Broader Context. Recall that the Laplacian Δ has eigenfunctions that can be expressed in terms of trigonometric functions and that the associated eigenvalues form an unbounded sequence.[2] Therefore, the Laplacian, regarded as an operator from $\mathcal{C}^2(\Omega)$ to $\mathcal{C}(\Omega)$, both spaces endowed with the sup-norm, is unbounded.[3] This property is inherited by discretizations in the sense that the size of the smallest interval containing all of the eigenvalues of the associated (stiffness) matrices will increase, without any upper bound, as the mesh parameter $h > 0$ decreases.

It may seem that using a different setting in which second order differential operators are coercive and bounded will resolve this matter. This is unfortunately not the case: Motivated by the weak/variational form of boundary value problems, one may consider $\nabla \cdot (K(x)\nabla u)$ as a mapping from $H_0^1(\Omega)$ onto its dual $H^{-1}(\Omega)$; see (1.4). Let $K(x)$ be a real symmetric (diffusion) tensor that is bounded and uniformly positive definite over the closure of the solution domain Ω. Within this setting, $\mathcal{A}$ not only becomes bounded, but is an homeomorphism. Still, a standard Galerkin finite

[2]For example, $\sin(i\pi x)\sin(j\pi y)$ is an eigenfunction with eigenvalue $-(i^2+j^2)\pi^2$ of the Laplacian on the unit square when homogeneous Dirichlet boundary conditions are employed.

[3]Recall that $\mathcal{C}^2(\Omega)$ is not complete, and therefore it is not a Hilbert space. The Hellinger–Toeplitz theorem does not apply.

element discretization yields symmetric positive definite stiffness matrices with the ratio κ of the largest to the smallest eigenvalue, called the condition number, growing proportionally to h^{-2}.

The numerical treatment of elliptic boundary value problems therefore becomes difficult because refining the discretization results in eigenvalues spread throughout a large interval. This led in the works of Axelsson, Evans, Concus, Golub, O'Leary, Meierink, van der Vorst, and many others to the development of preconditioners defined by, e.g., matrix splittings and incomplete Cholesky factorization techniques. This reduces the spreading of the eigenvalues, but the size of the smallest interval containing all of the eigenvalues of the preconditioned matrices will still increase, without any bound, as $h \to 0$.

In order to obtain h-robust bounds for all the eigenvalues, one can employ *operator preconditioning*: The preconditioner, on the infinite-dimensional Hilbert space level, is then defined in terms of a bounded linear mapping from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. For example, for second order elliptic PDEs, the inverse of the Laplacian, Δ^{-1}, is a prototypical operator preconditioner, which leads to our interest in the spectral properties of $\Delta^{-1}\nabla \cdot (K\nabla u)$ with suitable boundary conditions. In this case, not only do we obtain h-independent bounds for the eigenvalues on the discrete level, but also the entire spectrum, in the infinite-dimensional setting, can be characterized in terms of the spectral decomposition of K; see (1.7)–(1.8).

Theorem 1.1 can be illustrated by the following experiment. We consider three test problems with diagonal tensors (1.2) (i.e., $Q = I$) defined on the domain $\Omega \equiv (0,1) \times (0,1)$, where

$$\begin{aligned}
&(\text{P1}): \quad \kappa_1(x,y) = 1, && \kappa_2(x,y) = 10,\\
&(\text{P2}): \quad \kappa_1(x,y) = 1 + 0.5(x+y), && \kappa_2(x,y) = 10 - 0.5(x+y),\\
&(\text{P3}): \quad \kappa_1(x,y) = 1 + 3(x+y), && \kappa_2(x,y) = 10 - 2(x+y)
\end{aligned} \tag{2.1}$$

for $(x,y) \in \Omega$. We discretize the operators (1.3) and (1.4) using a uniform triangular mesh with piecewise linear discretization basis functions; see [9] for the scalar case analogy. Figure 1 presents the eigenvalues of the resulting generalized algebraic eigenvalue problem of size 381. We observe that the spectrum of the discretized problem not only covers the union of the ranges $\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})$, but in the case that $\kappa_1(\overline{\Omega})$ and $\kappa_2(\overline{\Omega})$ do not overlap, it surprisingly covers the whole interval (1.8).

Using the suggestive term *preconditioning* has led to some misinterpretations, because reducing the condition number of matrices arising from discretizations has become incorrectly associated with always leading to faster convergence behavior of Krylov subspace methods such as the conjugate gradient (CG) method. This flaw, ignoring the distribution of the eigenvalues between their maximum and minimum, has unfortunately become a common false wisdom spread throughout the literature, including many textbooks. In fact, the convergence behavior of CG is often determined by the *entire* spectral distribution functions of the linear systems involved; see, e.g., [11, 18]. Hence, the analysis presented in this paper can be employed to better understand the performance of CG when the inverse of the Laplacian (or some variant incorporating it) is applied as a preconditioner to solve discretized second order elliptic PDEs. Further details on this matter are presented in [9], which contains an instructive example analyzing the behavior of CG applied to a model problem. Also, constant-coefficient preconditioners may be of particular interest when the isogeometric analysis (IgA) approach is employed to discretize both PDEs and the computational domains involved in terms of B-splines [12, 13, 25]. For examples that use ideas

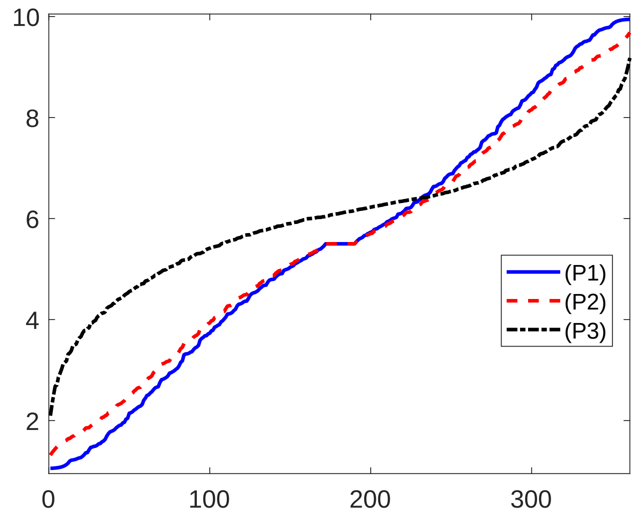

Fig. 1 *Eigenvalues of the discretized problems* (P1)–(P3)*, defined in* (2.1)*, spread over the entire interval* $[1, 10]$*; the ranges of entries of the diagonal tensor are as follows:* (P1)*:* $\kappa_1(\overline{\Omega}) = 1$, $\kappa_2(\overline{\Omega}) = 10$*;* (P2)*:* $\kappa_1(\overline{\Omega}) = [1, 2]$, $\kappa_2(\overline{\Omega}) = [9, 10]$*;* (P3)*:* $\kappa_1(\overline{\Omega}) = [1, 7]$, $\kappa_2(\overline{\Omega}) = [6, 10]$*. Horizontal axis: the indices of the increasingly ordered eigenvalues. Vertical axis: the size of the eigenvalues.*

inspired by [9] for developing a complementary approach that has now been applied to engineering problems, we refer the reader to the recent works [16, 22, 17, 15].

This paper is organized as follows. For clarity of exposition, we restrict ourselves in sections 3 and 4 to problems with diagonal tensors. In section 3 we present auxiliary lemmas generalizing, step by step, the results in [20]. Section 4 contains the proof of the main result for problems with diagonal tensors, and in section 5 we generalize the lemmas from previous sections to nondiagonal symmetric tensors and give the proof of the main result, Theorem 1.1. In section 6 we comment on problems with homogeneous Neumann boundary conditions. The numerical experiments in section 7 illustrate the results of the analysis. Furthermore, our results concerning $\Delta^{-1}\nabla \cdot (K\nabla u)$ lead us to the question of whether, in a more general operator theoretical setting, the entire spectrum of a preconditioned operator can be approximated by the eigenvalues of its discretizations, which is analyzed in [10]. The answer to this question, known in spectral approximation theory as lower semicontinuity of the operator spectrum, is "yes." We discuss some of the results presented in [10] in section 8. This further shows the importance of determining the spectrum of infinitely dimensional problems in order to forecast the behavior of iterative schemes applied to discretized operator preconditioned boundary value problems. The text closes with a brief discussion of some open problems in section 9.

Since $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all u, $v \in H_0^1(\Omega)$, which is a consequence of the symmetry of the tensor K, the preconditioned operator (1.5) is self-adjoint with respect to the inner product associated with the Laplacian:

$$(u, v)_{\mathcal{L}} \equiv \langle \mathcal{L}u, v\rangle = \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in H_0^1(\Omega), \tag{2.2}$$

$$(\mathcal{L}^{-1}\mathcal{A}u, v)_{\mathcal{L}} = \langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle = (\mathcal{L}^{-1}\mathcal{A}v, u)_{\mathcal{L}}. \tag{2.3}$$

Consequently, $\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) \subset \mathbb{R}$. The inner product (2.2) defines the norm

$$\|u\|_{\mathcal{L}}^2 \equiv (u,u)_{\mathcal{L}} = \langle \mathcal{L}u, u\rangle = \int_\Omega \|\nabla u\|^2 = \int_\Omega \|u_x\|_2^2 + \|u_y\|_2^2, \quad u \in H_0^1(\Omega),$$

used in the proofs below.

3. Auxiliary Results.

We will start by considering diagonal tensors, i.e.,

$$K(x,y) = \begin{bmatrix} \kappa_1(x,y) & 0 \\ 0 & \kappa_2(x,y) \end{bmatrix}. \tag{3.1}$$

This will allow us to explain with full clarity the main difference between the scalar case studied in [9, 20] and the tensor case analyzed in this paper.

3.1. Function Values at Points of Continuity Belong to the Spectrum.

The following lemma generalizes statement (a) in Theorem 3.1 in [20].

LEMMA 3.1. *Assume that K is a diagonal tensor, where the entries κ_1 and κ_2 are bounded and Lebesgue integrable functions on Ω. The following holds for $i = 1, 2$: If κ_i is continuous at $(x_0, y_0) \in \Omega$, then*

$$\kappa_i(x_0,y_0) \in \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}).$$

Proof. Assume that κ_1 is continuous at (x_0, y_0), and let

$$\lambda \equiv \kappa_1(x_0,y_0).$$

We will construct parametrized functions v_r and $u_r = (\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A})v_r$ such that

$$\lim_{r\to 0} \|v_r\|_{\mathcal{L}} \neq 0 \quad \text{and} \quad \lim_{r\to 0} \|u_r\|_{\mathcal{L}} = 0, \tag{3.2}$$

which is not possible if $\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A}$ has a bounded inverse: $v_r = (\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A})^{-1}u_r$ and $\lim_{r\to 0}\|u_r\|_{\mathcal{L}} = 0$ imply that $\lim_{r\to 0}\|v_r\|_{\mathcal{L}} = 0$. (The norm $\|\cdot\|_{\mathcal{L}}$ is the norm induced by the inner product (2.2).)

The functions v_r can be constructed, e.g., in the following way. Consider, for a sufficiently small $r > 0$, the following closed neighborhood of the point (x_0, y_0):

$$R_r = [x_0 - r^2, x_0 + r^2] \times [y_0 - r, y_0 + r] \subset \Omega. \tag{3.3}$$

For $(x,y) \in R_r$, define

$$v_r(x,y) = \sqrt{r}\min\left\{1 - \tfrac{|x-x_0|}{r^2}, \tfrac{1}{r} - \tfrac{|y-y_0|}{r^2}\right\}, \tag{3.4}$$

and $v_r(x,y) = 0$ otherwise. It can be verified (see Appendix A) that

$$\begin{aligned} 4 - 4r \le \|(v_r)_x\|_{L^2(\Omega)}^2 &\le 4, \\ \|(v_r)_y\|_{L^2(\Omega)}^2 &\le 4r. \end{aligned} \tag{3.5}$$

Consequently,

$$\lim_{r\to 0}\|v_r\|_{\mathcal{L}} = \lim_{r\to 0}\left(\|(v_r)_x\|_{L^2(\Omega)}^2 + \|(v_r)_y\|_{L^2(\Omega)}^2\right)^{1/2} = 2. \tag{3.6}$$

Considering that

$$u_r = (\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A})v_r, \quad \text{i.e.,} \quad \mathcal{L}u_r = (\lambda\mathcal{L} - \mathcal{A})v_r, \tag{3.7}$$

we get

$$\begin{aligned}\|u_r\|_{\mathcal{L}}^2 = \langle \mathcal{L}u_r, u_r\rangle &= \langle(\lambda\mathcal{L} - \mathcal{A})v_r, u_r\rangle \\ &= \int_\Omega (\lambda I - K)\nabla v_r \cdot \nabla u_r \\ &\le \left(\int_\Omega |(\lambda I - K)\nabla v_r|^2\right)^{1/2} \|u_r\|_{\mathcal{L}}.\end{aligned}$$

Using the fact that $\operatorname{supp}(v_r) = R_r$ and (3.5),

$$\begin{aligned}\|u_r\|_{\mathcal{L}}^2 &\le \|(\lambda - \kappa_1)(v_r)_x\|_{L^2(\Omega)}^2 + \|(\lambda - \kappa_2)(v_r)_y\|_{L^2(\Omega)}^2 \\ &\le 4 \sup_{(x,y)\in R_r} |\kappa_1(x_0, y_0) - \kappa_1(x, y)|^2 + 4r(\|\kappa_1\|_{L^\infty(\Omega)} + \|\kappa_2\|_{L^\infty(\Omega)})^2,\end{aligned}$$

and from the continuity of $\kappa_1(x, y)$ at (x_0, y_0),

$$\lim_{r\to 0} \|u_r\|_{\mathcal{L}} = 0. \tag{3.8}$$

From (3.6) and (3.8) we conclude that we can construct functions v_r and $u_r = (\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A})v_r$ such that (3.2) holds. We conclude that $\kappa_1(x_0, y_0)\mathcal{I} - \mathcal{L}^{-1}\mathcal{A}$ cannot have a bounded inverse.

The proof that $\kappa_2(x_0, y_0)$ belongs to the spectrum if κ_2 is continuous at (x_0, y_0) is trivially analogous. ∎

If $\kappa_i \in \mathcal{C}(\Omega)$, $i = 1, 2$, then Lemma 3.1 gives a diagonal tensor case analogy of Theorem 3.1, statement (b), in [20]. As is shown next, in the tensor case the spectrum of the preconditioned operator $\mathcal{L}^{-1}\mathcal{A}$ can, however, also contain numbers that do not belong to any of the individual ranges of the functions κ_1 and κ_2.

3.2. Disjoint Ranges Extend the Spectrum. An unexpected case occurs when the ranges of κ_1 and κ_2 are disjoint:

$$\kappa_1(\overline{\Omega}) \cap \kappa_2(\overline{\Omega}) = \emptyset.$$

We begin by presenting the following facts that will be used in the proofs.

3.2.1. Dirichlet Problem for the Wave Equation. Note that for any integer n,

$$\phi(x, y) = \sin(n\pi c l^{-1}(y - y_0)) \sin(n\pi l^{-1}(x - x_0)) \tag{3.9}$$

solves the following Dirichlet problem for the wave equation:

$$\begin{aligned}\phi_{yy} &= c^2\phi_{xx} \quad \text{in } \Sigma_l, \\ \phi &= 0 \quad \text{on } \partial\Sigma_l,\end{aligned} \tag{3.10}$$

where l is a positive constant which determines the size of the solution domain

$$\Sigma_l = (x_0, x_0 + l) \times (y_0, y_0 + l/c)$$

and $c > 0$ is arbitrary. We conclude that this Dirichlet problem has infinitely many nontrivial solutions. It is also clear that Σ_l can be made as small as needed by choosing $l > 0$ sufficiently small.

3.2.2. Tensors Constant on an Open Subdomain. Consider the generalized eigenvalue problem (1.10) with a diagonal tensor $K(x, y)$ (3.1) that is constant on an open subdomain $S \subset \Omega$. Then we get the following lemma.

LEMMA 3.2. *Consider a diagonal tensor* (3.1)*, where the bounded and Lebesgue integrable functions κ_i, $i = 1, 2$, are constant on an open subdomain $S \subset \Omega$. Assuming that*

$$\sup_{(x,y)\in\Omega} \kappa_1(x, y) < \inf_{(x,y)\in\Omega} \kappa_2(x, y), \tag{3.11}$$

the following closed interval belongs to the spectrum of $\mathcal{L}^{-1}\mathcal{A}$:

$$\left[\sup_{(x,y)\in\Omega} \kappa_1(x, y), \inf_{(x,y)\in\Omega} \kappa_2(x, y) \right] \subset \operatorname{sp}(\mathcal{L}^{-1}\mathcal{A}). \tag{3.12}$$

The analogous statement obviously holds by interchanging the roles of κ_1 and κ_2 in (3.11) *and* (3.12)*.*

Proof. Consider an arbitrary fixed point $(x_0, y_0) \in S$. For any fixed $c > 0$, there exists $l > 0$ such that

$$\Sigma_l \equiv (x_0, x_0 + l) \times (y_0, y_0 + l/c) \subset S.$$

Since $K(x, y)$ is constant on Σ_l, we can rewrite (1.10) as

$$(\lambda - \overline{k}_1)v_{xx} + (\lambda - \overline{k}_2)v_{yy} = 0 \quad \text{in } \Sigma_l, \tag{3.13}$$

where $\overline{k}_1$ and $\overline{k}_2$ are constants, and

$$K(x, y) = \begin{bmatrix} \overline{k}_1 & 0 \\ 0 & \overline{k}_2 \end{bmatrix}, \quad (x, y) \in \Sigma_l.$$

Consider an arbitrary λ in the interval $(\overline{k}_1, \overline{k}_2)$. Then (3.13) represents, with

$$c^2 = \frac{\lambda - \overline{k}_1}{\overline{k}_2 - \lambda} > 0,$$

the wave equation (3.10). Taking any nontrivial solution ϕ of (3.10), the function v defined on Ω as

$$v(x, y) = \begin{cases} \phi(x, y), & (x, y) \in \Sigma_l, \\ 0, & (x, y) \notin \Sigma_l, \end{cases}$$

solves the weak form of the generalized eigenvalue problem (1.10): Using

$$0 = \langle \mathcal{A}v, \psi \rangle - \lambda \langle \mathcal{L}v, \psi \rangle = \langle \mathcal{L}(\mathcal{L}^{-1}\mathcal{A}v - \lambda v), \psi \rangle \quad \text{for all } \psi \in H_0^1(\Omega),$$

we get $\mathcal{L}^{-1}\mathcal{A}v = \lambda v$, i.e., λ is an eigenvalue and v is an eigenfunction of the preconditioned operator $\mathcal{L}^{-1}\mathcal{A}$. We conclude that $(\overline{k}_1, \overline{k}_2) \subset \operatorname{sp}(\mathcal{L}^{-1}\mathcal{A})$.

Since, by construction,

$$\overline{k}_1 \leq \sup_{(x,y)\in\Omega} \kappa_1(x, y) < \inf_{(x,y)\in\Omega} \kappa_2(x, y) \leq \overline{k}_2, \tag{3.14}$$

it remains to prove that if the equality is attained on any side of (3.14), then the associated $\overline{k}_i$, $i = 1$ and/or $i = 2$, also belongs to the spectrum of $\mathcal{L}^{-1}\mathcal{A}$. But this is trivially true using Lemma 3.1 because $\overline{k}_i$ is a function value of $\kappa_i(x, y)$ at Σ_l, where κ_i is constant and therefore continuous. ∎

Lemma 3.2 shows that, under the given assumptions, the whole closed interval determined by the extremal points of the disjoint ranges of κ_1 and κ_2 belongs to the spectrum of the preconditioned operator $\mathcal{L}^{-1}\mathcal{A}$. Moreover, each inner point of this interval is an eigenvalue. Please note that it is not assumed here that K is continuous throughout the closure $\overline{\Omega}$ and that the subdomain S is of an arbitrarily small size.

3.2.3. Tensors Continuous at Least at a Single Point. The following lemma further refines the assumptions under which the statement of Lemma 3.2 holds.

LEMMA 3.3. *Assume that the diagonal tensor* (3.1) *with the bounded and Lebesgue integrable functions* κ_i*,* $i = 1, 2$*, is continuous (at least) at a single point in* Ω*. If*

$$\sup_{(x,y)\in\Omega} \kappa_1(x, y) < \inf_{(x,y)\in\Omega} \kappa_2(x, y), \tag{3.15}$$

then the following closed interval belongs to the spectrum of $\mathcal{L}^{-1}\mathcal{A}$*:*

$$\left[\sup_{(x,y)\in\Omega} \kappa_1(x, y), \ \inf_{(x,y)\in\Omega} \kappa_2(x, y) \right] \subset \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}). \tag{3.16}$$

The analogous statement obviously holds by interchanging the roles of κ_1 *and* κ_2 *in* (3.15) *and* (3.16)*.*

Proof. We will prove the statement by contradiction. Consider

$$\lambda \in \left[\sup_{(x,y)\in\Omega} \kappa_1(x, y), \ \inf_{(x,y)\in\Omega} \kappa_2(x, y) \right]$$

such that $\lambda \notin \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A})$, i.e., such that the operator $\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I}$ has a bounded inverse.

Let $(x_0, y_0) \in \Omega$ be the point of continuity of the tensor $K(x, y)$. Applying Lemma 3.2 to the preconditioned operator $\mathcal{L}^{-1}\mathcal{A}_l$, where $\mathcal{A}_l$ is defined for any sufficiently small l by

$$\langle \mathcal{A}_l\phi, \psi\rangle \equiv \int_\Omega K_l \nabla\phi \cdot \nabla\psi, \quad \phi, \psi \in H_0^1(\Omega),$$

and $K_l(x, y)$ is a local modification of K,

$$K_l(x, y) \equiv \begin{cases} K(x_0, y_0), & (x, y) \in S_l, \\ K(x, y), & (x, y) \in \Omega \setminus S_l, \end{cases}$$
$$S_l = (x_0, x_0 + l) \times (y_0, y_0 + l),$$

yields that

$$\lambda \in \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}_l). \tag{3.17}$$

On the other hand, since we assume that $\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I}$ is invertible,

$$\begin{aligned} \mathcal{L}^{-1}\mathcal{A}_l - \lambda\mathcal{I} &= (\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I}) + (\mathcal{L}^{-1}\mathcal{A}_l - \mathcal{L}^{-1}\mathcal{A}) \\ &= (\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})[\mathcal{I} + (\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})^{-1}\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})]. \end{aligned}$$

In Appendix B we prove that for sufficiently small $l > 0$,

$$\|(\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})^{-1}\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_\mathcal{L} < 1, \tag{3.18}$$

and a Neumann series argument therefore ensures that $\mathcal{L}^{-1}\mathcal{A}_l - \lambda\mathcal{I}$ has a bounded inverse. Consequently, $\lambda \notin \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}_l)$, which contradicts (3.17). (Inequality (3.18) holds due to the assumption that $\lambda \notin \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A})$ and due to the continuity of $K(x,y)$ at the point (x_0, y_0). See Appendix B for further details.) ∎

It is worth noting that the statement of Lemma 3.3 requires continuity of the tensor K only at an arbitrary *single point* belonging to Ω.

4. Continuous Diagonal Tensors.

We first complement Lemma 3.1, and Theorem 3.1 in [20], by proving the "reverse inclusion."

4.1. The Spectrum Is a Subset of the Extremal Interval.

LEMMA 4.1. *Assume that the diagonal tensor* (3.1) *is continuous throughout the closure* $\overline{\Omega}$. *Then*

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) \subset \mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})). \tag{4.1}$$

Proof. Using the self-adjointness (2.3) of the operator $\mathcal{L}^{-1}\mathcal{A}$, we can take the standard results from the theory of self-adjoint operators (see, e.g., [8, section 6.5]) and conclude that the spectrum of $\mathcal{L}^{-1}\mathcal{A}$ is real and that

$$\begin{aligned}\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) &\subset \left[\inf_{u\in H_0^1(\Omega)} \frac{(\mathcal{L}^{-1}\mathcal{A}u,u)_{\mathcal{L}}}{(u,u)_{\mathcal{L}}}, \sup_{u\in H_0^1(\Omega)} \frac{(\mathcal{L}^{-1}\mathcal{A}u,u)_{\mathcal{L}}}{(u,u)_{\mathcal{L}}}\right] \\ &= \left[\inf_{u\in H_0^1(\Omega)} \frac{\langle\mathcal{A}u,u\rangle}{\langle\mathcal{L}u,u\rangle}, \sup_{u\in H_0^1(\Omega)} \frac{\langle\mathcal{A}u,u\rangle}{\langle\mathcal{L}u,u\rangle}\right]. \end{aligned} \tag{4.2}$$

Moreover, the endpoints of this interval are contained in the spectrum.

It remains to bound

$$\frac{\langle\mathcal{A}u,u\rangle}{\langle\mathcal{L}u,u\rangle} \tag{4.3}$$

in terms of the extreme values of the scalar functions κ_1 and κ_2. Since $u_x^2(x,y) \geq 0$ and $u_y^2(x,y) \geq 0$, we can bound (4.3) as follows:

$$\begin{aligned}\sup_{u\in H_0^1(\Omega)} \frac{\langle\mathcal{A}u,u\rangle}{\langle\mathcal{L}u,u\rangle} &= \sup_{u\in H_0^1(\Omega)} \frac{\int_\Omega K\nabla u\cdot\nabla u}{\int_\Omega \|\nabla u\|^2} = \sup_{u\in H_0^1(\Omega)} \frac{\int_\Omega \kappa_1 u_x^2 + \kappa_2 u_y^2}{\int_\Omega \|\nabla u\|^2} \\ &\leq \sup_{u\in H_0^1(\Omega)} \frac{\int_\Omega \sup_{(x,y)\in\Omega}\max_{i=1,2}\{\kappa_i(x,y)\}\,\|\nabla u\|^2}{\int_\Omega \|\nabla u\|^2} \\ &\leq \sup_{(x,y)\in\Omega}\max_{i=1,2}\{\kappa_i(x,y)\}. \end{aligned} \tag{4.4}$$

Similarly,

$$\inf_{u\in H_0^1(\Omega)} \frac{\langle\mathcal{A}u,u\rangle}{\langle\mathcal{L}u,u\rangle} \geq \inf_{(x,y)\in\Omega}\min_{i=1,2}\{\kappa_i(x,y)\}.$$

For $K(x,y)$ continuous on $\overline{\Omega}$, the infimum and supremum of its components $\kappa_1(x,y)$ and $\kappa_2(x,y)$ are attained. Please note that no assumption is made about the positive (negative) definiteness of K. ∎

We are now ready to prove Theorem 1.1 for continuous diagonal tensors.

4.2. Main Result—Diagonal Tensors.

THEOREM 4.2. *Consider an open and bounded Lipschitz domain* $\Omega \subset \mathbb{R}^2$*. If the diagonal tensor* (3.1) *is continuous throughout the closure* $\overline{\Omega}$*, then*

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) = \mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})).$$

Proof. Assume that the diagonal tensor $K(x,y)$ is continuous throughout $\overline{\Omega}$. Then, by Lemmas 3.1 and 3.3,

$$\mathrm{Conv}(\kappa_1(\Omega) \cup \kappa_2(\Omega)) \subset \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}),$$

and due to the continuity of $K(x,y)$ and the fact that $\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A})$ is a closed set (see, e.g., [24]),

$$\mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})) \subset \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}).$$

Finally, by Lemma 4.1,

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) \subset \mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})),$$

which gives the statement. □

5. Proof of Theorem 1.1. It remains to revisit and complete the arguments given above for the general self-adjoint operator in (1.4). Consider the general symmetric tensor

$$K(x,y) = \begin{bmatrix} k_1(x,y) & k_3(x,y) \\ k_3(x,y) & k_2(x,y) \end{bmatrix}, \tag{5.1}$$

where k_1, k_2, and k_3 are bounded and Lebesgue integrable functions defined on Ω, with the spectral decomposition

$$K(x,y) = Q(x,y) \begin{bmatrix} \kappa_1(x,y) & 0 \\ 0 & \kappa_2(x,y) \end{bmatrix} Q^T(x,y); \tag{5.2}$$

see (1.2).

The structure of the proof of Theorem 1.1 is fully analogous to the proof of Theorem 4.2 formulated for diagonal tensors. We will now restate the associated lemmas for the general case and comment on the technical differences that must be considered.

For convenience, we will use, when appropriate, the column vector notation

$$\mathbf{w} = (x,y)^T, \quad (x,y) \in \Omega,$$

and for any function f defined on Ω its gradient ∇f will be considered as a column vector.

LEMMA 5.1 (see Lemma 3.1). *Consider the symmetric tensor* (5.1) *with the spectral decomposition* (5.2)*. If the tensor* K *is continuous at* $(x_0,y_0) \in \Omega$*, then*

$$\kappa_i(x_0,y_0) \in \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}), \quad i = 1,2.$$

Proof. We will use the following notation for the spectral decomposition of $K(x,y)$ at the point of continuity (x_0,y_0):

$$K_0 \equiv K(x_0,y_0) = Q_0\Lambda_0 Q_0^T, \quad Q_0 \equiv Q(x_0,y_0), \quad Q_0^T Q_0 = I,$$
$$\Lambda_0 \equiv \Lambda(x_0,y_0) = \mathrm{diag}(\kappa_1(x_0,y_0), \kappa_2(x_0,y_0)).$$

Simple algebraic computations give that, for any $(x,y)\in\Omega$,

$$\kappa_1 = \tfrac{1}{2}(k_1+k_2+\sqrt{D}), \quad \kappa_2 = \tfrac{1}{2}(k_1+k_2-\sqrt{D}), \tag{5.3}$$

where $D = (k_1-k_2)^2 + 4k_3^2$. Therefore, at any point of continuity of the tensor $K(x,y)$, the functions $\kappa_1(x,y)$ and $\kappa_2(x,y)$ are also continuous.

For sufficiently small r, consider the closed neighborhood R_r defined in (3.3) and its counterpart defined as

$$S_r = \{Q_0\mathbf{z} \mid \mathbf{z}\in R_r\},$$

where the choice of r in (3.3) ensures that both $R_r \subset \Omega$ and $S_r\subset\Omega$. Consider the functions

$$\tilde{v}_r(\mathbf{w}) \equiv v_r(Q_0^T\mathbf{w}), \quad \mathbf{w}\in\Omega,$$

where v_r is defined in (3.4). Since $|\det Q| = 1$, the change of variables gives

$$\|\tilde{v}_r\|_{\mathcal{L}}^2 = \int_{S_r} \|\nabla\tilde{v}_r(\mathbf{w})\|^2 d\mathbf{w} = \int_{R_r} \|\nabla v_r(\mathbf{z})\|^2 d\mathbf{z} = \|v_r\|_{\mathcal{L}}^2 \tag{5.4}$$

and, from (3.6),

$$\lim_{r\to 0} \|\tilde{v}_r\|_{\mathcal{L}} = 2 \neq 0. \tag{5.5}$$

Analogously to (3.7) we consider

$$u_r \equiv (\lambda\mathcal{I} - \mathcal{L}^{-1}\mathcal{A})\tilde{v}_r, \quad \lambda\equiv\kappa_1(x_0,y_0),$$

with the norm

$$\|u_r\|_{\mathcal{L}}^2 = \int_\Omega (\lambda I - K)\nabla\tilde{v}_r\cdot\nabla u_r \tag{5.6}$$
$$= \int_{S_r} (\lambda I - K_0)\nabla\tilde{v}_r\cdot\nabla u_r + \int_{S_r} (K_0 - K)\nabla\tilde{v}_r\cdot\nabla u_r. \tag{5.7}$$

Our goal is to show that if $\lambda\notin \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A})$, then $\lim_{r\to 0}\|u_r\|_{\mathcal{L}} = 0$, which contradicts (5.5). Concerning the second integral in (5.7),

$$\int_{S_r} (K_0 - K)\nabla\tilde{v}_r\cdot\nabla u_r \le \sup_{\mathbf{w}\in S_r} \|K_0 - K(\mathbf{w})\|\,\|\tilde{v}_r\|_{\mathcal{L}}\,\|u_r\|_{\mathcal{L}}.$$

Using the continuity of $K(x,y)$ at the point (x_0,y_0) and the fact that $\|\tilde{v}_r\|_{\mathcal{L}}\|u_r\|_{\mathcal{L}}$ is bounded, the second integral on the right-hand side of (5.7) vanishes as $r\to 0$. For the remaining term in (5.7), we find that

$$\begin{aligned}\int_{S_r} (\lambda I - K_0)\nabla\tilde{v}_r\cdot\nabla u_r &= \int_{S_r} Q_0(\lambda I - \Lambda_0)Q_0^T\nabla\tilde{v}_r\cdot\nabla u_r \\ &\le \left(\int_{S_r} \|Q_0(\lambda I - \Lambda_0)Q_0^T\nabla\tilde{v}_r\|^2\right)^{1/2} \|u_r\|_{\mathcal{L}}.\end{aligned}$$

Applying the chain rule gives $\nabla\tilde{v}_r(\mathbf{w}) = Q_0\nabla v_r(Q_0^T\mathbf{w}) = Q_0\nabla v_r(\mathbf{z})$, which together with the orthogonality of Q_0 gives (considering $\lambda = \kappa_1(x_0, y_0)$)

$$\begin{aligned}\int_{S_r} \|Q_0(\lambda I - \Lambda_0)Q_0^T\nabla\tilde{v}_r\|^2 &= \int_{S_r} \|(\lambda I - \Lambda_0)\nabla v_r(Q_0^T\mathbf{w})\|^2 \\ &= \int_{R_r} \|(\lambda I - \Lambda_0)\nabla v_r(\mathbf{z})\|^2 \\ &= \int_{R_r} \|(\lambda - \kappa_2(x_0, y_0))(v_r)_y(\mathbf{z})\|^2 \\ &\le |\lambda - \kappa_2(x_0, y_0)|\, \|(v_r)_y\|^2_{L^2(\Omega)},\end{aligned}$$

where the upper bound vanishes as $r \to 0$ due to (3.5).

The proof that $\kappa_2(x_0, y_0)$ belongs to the spectrum of the preconditioned operator, provided that the assumptions of the lemma hold, is trivially analogous. ∎

The remaining part of the proof of Theorem 1.1 is a straightforward extension of the analysis presented in section 4.

LEMMA 5.2 (see Lemma 3.2). *Consider a symmetric tensor* (5.1) *with bounded and Lebesgue integrable functions k_i, $i = 1, 2, 3$, which are constant on an open subdomain $S \subset \Omega$. Assuming that*

$$\sup_{(x,y)\in\Omega} \kappa_1(x, y) < \inf_{(x,y)\in\Omega} \kappa_2(x, y), \tag{5.8}$$

the following closed interval belongs to the spectrum of $\mathcal{L}^{-1}\mathcal{A}$:

$$\left[\sup_{(x,y)\in\Omega} \kappa_1(x, y),\ \inf_{(x,y)\in\Omega} \kappa_2(x, y)\right] \subset \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}). \tag{5.9}$$

The analogous statement obviously holds by interchanging the roles of κ_1 and κ_2 in (5.8) *and* (5.9).

Proof. Since $K(x, y)$ and its spectral decomposition $K = \bar{Q}\bar{\Lambda}\bar{Q}^T$ are constant on S, the change of variables $\mathbf{w} = \bar{Q}\mathbf{z}$ transforms the eigenvalue problem (1.10) in the subdomain S to the form

$$\nabla_{\mathbf{z}} \cdot (\bar{\Lambda}\nabla_{\mathbf{z}} v) = \lambda\Delta_{\mathbf{z}} v \quad \text{in } R = \{\bar{Q}^T\mathbf{w} \,|\, \mathbf{w} \in S\},$$

where the diagonal tensor $\bar{\Lambda}$ is constant. Employing the argument used to prove Lemma 3.2 finishes the proof. ∎

LEMMA 5.3 (see Lemma 3.3). *Assume that the symmetric tensor* (5.1) *with the bounded and Lebesgue integrable functions k_i, $i = 1, 2, 3$, is continuous at least at a single point in Ω. Assuming that*

$$\sup_{(x,y)\in\Omega} \kappa_1(x, y) < \inf_{(x,y)\in\Omega} \kappa_2(x, y), \tag{5.10}$$

the following closed interval belongs to the spectrum of $\mathcal{L}^{-1}\mathcal{A}$:

$$\left[\sup_{(x,y)\in\Omega} \kappa_1(x, y),\ \inf_{(x,y)\in\Omega} \kappa_2(x, y)\right] \subset \mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}). \tag{5.11}$$

The analogous statement obviously holds by interchanging the roles of κ_1 and κ_2 in (5.10) *and* (5.11).

Proof. The proof is fully analogous to the proof of Lemma 3.3. □

LEMMA 5.4 (see Lemma 4.1). *Let the symmetric tensor* (5.1) *be continuous throughout the closure* $\overline{\Omega}$. *Then*

$$\mathrm{sp}(\mathcal{L}^{-1}\mathcal{A}) \subset \mathrm{Conv}(\kappa_1(\overline{\Omega}) \cup \kappa_2(\overline{\Omega})).$$

Proof. The proof is analogous to the proof of Lemma 4.1 with the argument used in the derivation of (4.4) now written in the form

$$K\nabla u \cdot \nabla u = \Lambda Q^T \nabla u \cdot Q^T \nabla u \leq \sup_{\mathbf{w}\in\Omega} \max_{i=1,2} \{\kappa_i(\mathbf{w})\} \|Q^T \nabla u\|^2.$$

Due to the orthogonality of Q we get

$$\int_\Omega K\nabla u \cdot \nabla u \leq \sup_{\mathbf{w}\in\Omega} \max_{i=1,2} \{\kappa_i(\mathbf{w})\} \int_\Omega \|\nabla u\|^2$$

and, similarly,

$$\inf_{\mathbf{w}\in\Omega} \max_{i=1,2} \{\kappa_i(\mathbf{w})\} \int_\Omega \|\nabla u\|^2 \leq \int_\Omega K\nabla u \cdot \nabla u. \qquad \square$$

The proof of Theorem 1.1 is completed by the combination of Lemmas 5.1 to 5.4; see the proof of Theorem 4.2.

6. Neumann Boundary Conditions.

Theorem 1.1 also holds for the spectrum of the preconditioned operator $\mathcal{L}^{-1}\mathcal{A}$ with homogeneous Neumann boundary conditions. Instead of $H_0^1(\Omega)$, we now employ the space

$$V = \left\{ v \in H^1(\Omega) |, \int_\Omega v = 0 \right\}$$

with the operators $\mathcal{L}, \mathcal{A} : V \to V^\#$ defined analogously to those above (see (1.3) and (1.4)):

$$\langle \mathcal{L}\phi, \psi \rangle = \int_\Omega \nabla\phi \cdot \nabla\psi, \quad \phi, \psi \in V,$$
$$\langle \mathcal{A}\phi, \psi \rangle = \int_\Omega K\nabla\phi \cdot \nabla\psi, \quad \phi, \psi \in V,$$

where $\mathcal{L}$ has a bounded inverse operator; see, e.g., [19, Example 7.2.2, page 117]. We still use the $\|\cdot\|_{\mathcal{L}}$-norm; see (2.2). For the Neumann problem, the functions v_r defined as in (3.4), and the solutions ϕ of the wave equation defined as in (3.9) must be modified to

$$v_r - \int_\Omega v_r \quad \text{and} \quad \phi - \int_{\Sigma_l} \phi,$$

respectively. The rest will follow in an analogous way to the analysis presented in this paper. The associated generalized PDE eigenvalue problem in this context reads

$$\begin{aligned} \nabla \cdot (K\nabla u) &= \lambda \Delta u \quad && \text{for } (x,y) \in \Omega, \\ (K - \lambda I)\nabla u \cdot \mathbf{n} &= 0 && \text{for } (x,y) \in \partial\Omega, \end{aligned} \tag{6.1}$$

where $\mathbf{n}$ denotes the outwards pointing unit normal vector of $\partial\Omega$.

7. Numerical Experiments. In this section our theoretical results will be illuminated by numerical experiments in which the matrices $\mathbf{A_n}$ and $\mathbf{L_n}$ are constructed using FEniCS [1]. The eigenvalues of the resulting generalized algebraic eigenvalue problem (1.9), i.e., the eigenvalues of the discretized preconditioned operator represented in the matrix form by $\mathbf{L_n}^{-1}\mathbf{A_n}$, are computed and visualized with MATLAB.[4] If not specified otherwise, we consider the domain $\Omega \equiv (0,1) \times (0,1)$, and a uniform triangular mesh with piecewise linear discretization basis functions is used.

The examples in section 2 concern diagonal positive definite tensors. We first complement this by performing experiments with nondiagonal indefinite tensors (5.1). We consider three test problems with zero Dirichlet boundary conditions and with the following entries in the symmetric tensor (5.1):

$$\begin{aligned}
&(\text{P4}): \quad k_1(x,y) = 5,\ k_2(x,y) = -5,\ k_3(x,y) = 0,\\
&(\text{P5}): \quad k_1(x,y) = 3,\ k_2(x,y) = -3,\ k_3(x,y) = 4,\\
&(\text{P6}): \quad k_1(x,y) = 3e^{-3(|x-0.5|+|y-0.5|)},\ k_2(x,y) = -k_1,\ k_3(x,y) = 4\cos(\tfrac{\pi(x+y-1)}{2}).
\end{aligned}$$

Using (5.3) gives for problems (P4) and (P5) that $\kappa_1(x,y) = -5$ and $\kappa_2(x,y) = 5$. Furthermore, for problem (P6), formula (5.3) yields

$$\kappa_{1,2}(x,y) = \pm\sqrt{k_1^2 + k_3^2} = \pm\sqrt{9e^{-6(|x-0.5|+|y-0.5|)} + 16\cos^2(\tfrac{\pi(x+y-1)}{2})},$$

such that $\kappa_1(\overline{\Omega}) = -\kappa_2(\overline{\Omega}) = [3e^{-3}, 5]$. As in Figure 1, the spectra visualized in Figure 2 spread over the entire interval (1.8) defined by the nonoverlapping ranges $\kappa_1(\overline{\Omega})$ and $\kappa_2(\overline{\Omega})$. Refining the mesh gives better approximations of the endpoints of the interval $[-5, 5]$. The fact that the tensor (5.1) is not diagonal has no qualitative effect on the observed experimental data. We will therefore only consider diagonal tensors in what follows.

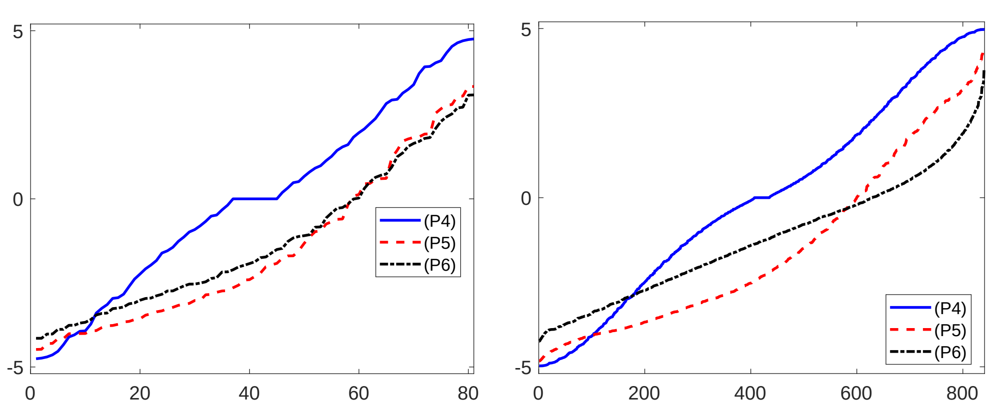

Fig. 2 *Spectra of the discretized test problems* (P4), (P5), *and* (P6) *for* $N = 81$ *(left) and* $N = 841$ *(right) degrees of freedom. Horizontal axis: the indices of the increasingly ordered eigenvalues. Vertical axis: the size of the eigenvalues.*

The left part of Figure 3 shows numerical results computed with homogeneous Neumann boundary conditions (see section 6). The results with zero Dirichlet

[4] FEniCS version 2017.2.0 [1] and MATLAB version 9.5.0 (R2018b).

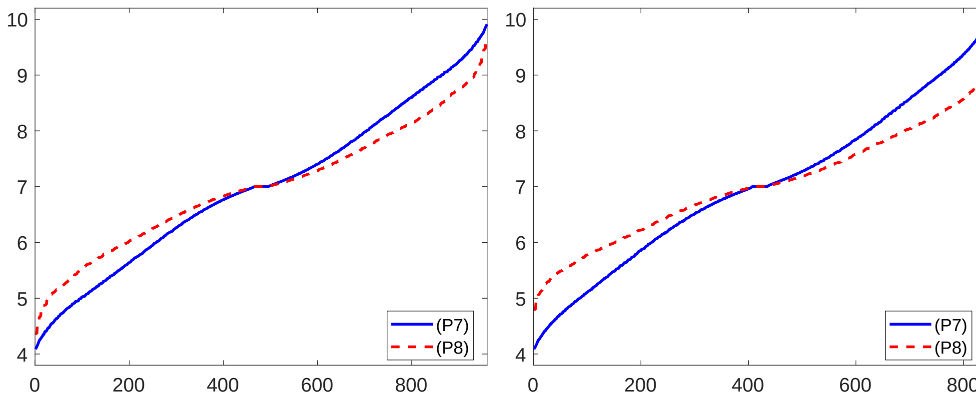

Fig. 3 *Spectra of the discretized test problems* (P7) *and* (P8) *with zero Neumann boundary conditions (left) and zero Dirichlet boundary conditions (right).*

boundary conditions are, for comparison, presented in the right part of Figure 3. We consider two test problems with the diagonal tensor (3.1) defined by

$$\begin{aligned} &(\text{P7}): \quad \kappa_1(x,y) = 10 - f(x,y), \quad \kappa_2(x,y) = 4 + f(x,y), \\ &(\text{P8}): \quad \kappa_1(x,y) = 8 + f(x,y), \quad \kappa_2(x,y) = 6 - f(x,y), \end{aligned} \tag{7.1}$$

where

$$f(x,y) = 4((x-0.5)^2 + (y-0.5)^2)$$

is chosen such that, for both problems, $\kappa_1(\overline{\Omega}) = [8, 10]$ and $\kappa_2(\overline{\Omega}) = [4, 6]$. Note that these intervals do not overlap. The minimum (respectively, maximum) of the interval $[4, 10]$ is obtained by the function $\kappa_1(x,y)$ (respectively, $\kappa_2(x,y)$) in the interior of the solution domain for problem (P7), while for problem (P8) the endpoints of this interval are attained on the boundary $\partial\Omega$. In the latter case the endpoints of the interval $[4, 10]$ are better approximated for the problem with Neumann boundary conditions. Similar behavior was also observed for other test cases.

Numerical results for nonconvex domains are presented in Figure 4. We used the diagonal tensor (3.1) with

$$(\text{P9}): \quad \kappa_1(x,y) = 6 - 3e^{-3(|x-0.8|+|y-0.8|)}, \quad \kappa_2(x,y) = 6 + 3e^{-3(|x-0.2|+|y-0.2|)}$$

and the L-shaped domains $\Omega_1 = (0,1)^2 \setminus (0,0.6)^2$ and $\Omega_2 = (0,1)^2 \setminus (0.4,1)^2$; see the illustration in the left part of Figure 4. We employed zero Dirichlet boundary conditions. The function $\kappa_1(x,y)$ (respectively, $\kappa_2(x,y)$) has its minimum (respectively, maximum) at the point $[0.8, 0.8]$ (respectively, $[0.2, 0.2]$), which is outside the domain Ω_2 (respectively, Ω_1). As a result, we observe in Figure 4 that the spectra of the discretized problems differ, depending on the ranges of functions $\kappa_1(x,y)$ and $\kappa_2(x,y)$ over $\overline{\Omega}_1$ and $\overline{\Omega}_2$.

Finally, we present in Figure 5 numerical results for 3D problems, which in this paper are not supported by rigorous proofs. The proofs were, however, provided after publication of the original SINUM paper by Ivana Pultarová in an unpublished note [21]. We consider the unit cube $\Omega \equiv (0,1)^3$, zero Dirichlet boundary conditions, and

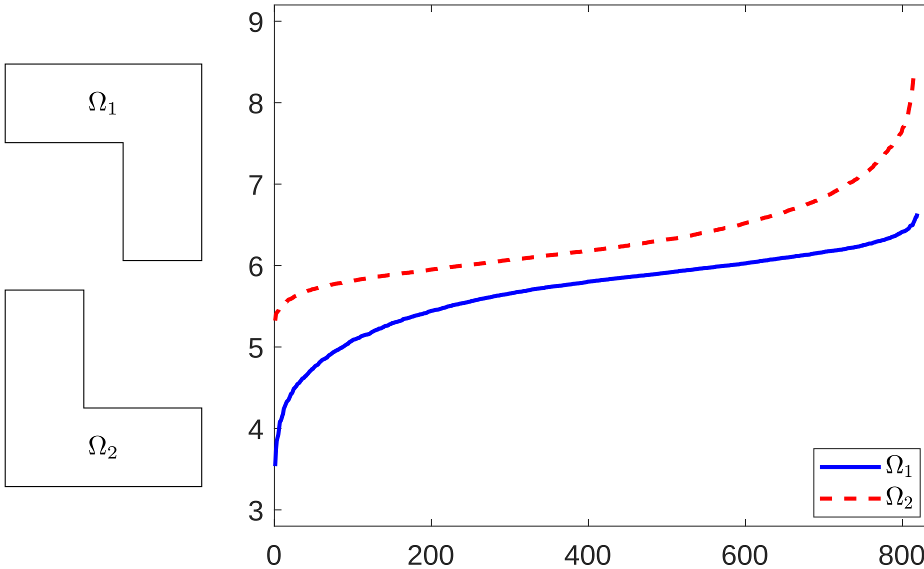

Fig. 4 *Left: illustration of the shapes of the domains Ω_1 and Ω_2. Right: spectra of the test problem* (P9) *associated with the domains Ω_1 and Ω_2. The ranges satisfy $\kappa_1(\overline{\Omega}_1) \subset [3,6]$ and $\kappa_2(\overline{\Omega}_1) \subset [6,7]$ for the domain Ω_1 and $\kappa_1(\overline{\Omega}_2) \subset [5,6]$ and $\kappa_2(\overline{\Omega}_2) \subset [6,9]$ for the domain Ω_2.*

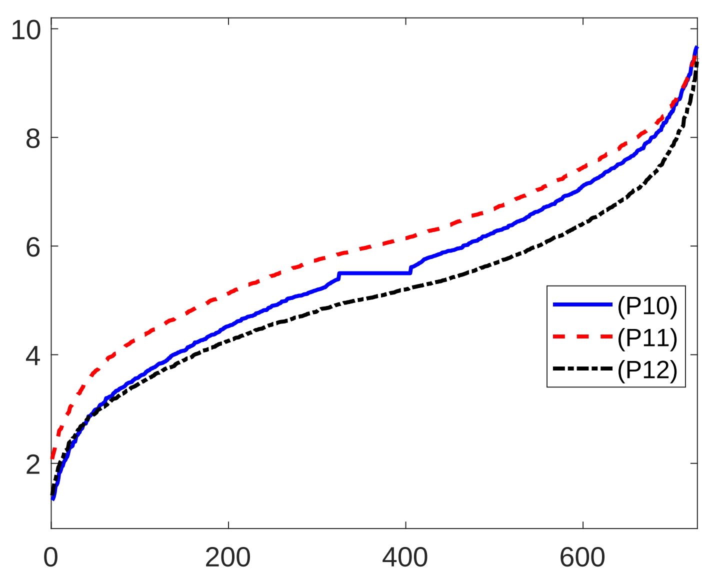

Fig. 5 *The spectra of the 3D test problems* (P10)–(P12) *spread over the entire interval $[1,10]$, while the ranges of the function entries of the diagonal tensors are as follows: isolated points $\kappa_1(\overline{\Omega}) = 1$, $\kappa_2(\overline{\Omega}) = 5.5$, and $\kappa_3(\overline{\Omega}) = 10$ for* (P10)*; nonoverlapping intervals $\kappa_1(\overline{\Omega}) = [1,2]$, $\kappa_2(\overline{\Omega}) = [4.5, 6.5]$, and $\kappa_3(\overline{\Omega}) = [9,10]$ for* (P11)*; and overlapping intervals $\kappa_1(\overline{\Omega}) = [1,5]$, $\kappa_2(\overline{\Omega}) = [4,6]$, and $\kappa_3(\overline{\Omega}) = [2,10]$ for* (P12)*.*

the diagonal tensor $K(x,y,z) = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3)$ defined as

$$
\begin{aligned}
&(\mathrm{P10}) : \kappa_1 = 1, \quad \kappa_2 = 5.5, \quad \kappa_3 = 10, \\
&(\mathrm{P11}) : \kappa_1 = 1 + \sin^2(x+y+z),\ \kappa_2 = 5.5 + \cos(\pi xyz),\ \kappa_3 = 10 - \cos^2(x+y+z), \\
&(\mathrm{P12}) : \kappa_1 = 1 + (x+y+z-1)^2,\ \kappa_2 = 4 + xy + z,\ \kappa_3 = 10 - 2(x+y+z-1)^2.
\end{aligned}
$$

This choice of test problems follows the same "pattern" as for the introductory experiments presented in section 2: The ranges of the functions $\kappa_i(x,y,z)$, $i = 1,2,3$, are for (P10) isolated points; they form nonoverlapping intervals for (P11) and overlapping intervals for (P12). As for the 2D test cases, we observe that the spectra of the discretized problems are spread over the entire interval $[1, 10]$, irrespective of whether or not the associated ranges overlap.

8. Convergence towards the Spectrum.

As observed above, our numerical experiments suggest that the eigenvalues of the discretized preconditioned operators cover, in the limit $h \to 0$, the whole interval (1.8). Here, h represents the mesh parameter. However, the convergence towards the individual points in this interval is not uniform. This observation, mentioned as a matter of further research in our original SINUM paper, was theoretically justified in the subsequent paper [10]. We will now state the result regarding this matter. The proof is presented in [10, section 4, Corollary 4.3] and addresses abstract operators defined on Hilbert spaces.

Consider an infinite-dimensional Hilbert space V, its dual $V^{\#}$, and bounded linear operators $\mathcal{A}, \mathcal{B} : V \to V^{\#}$ that are self-adjoint with respect to the duality pairing, and $\mathcal{B}$ is, in addition, coercive. Consider further a sequence of finite-dimensional subspaces $\{V_n\}$ of V, where n denotes the dimensionality, satisfying the standard approximation property

$$\lim_{n\to\infty} \inf_{v \in V_n} \|w - v\| = 0 \quad \text{for all } w \in V,$$

which typically yields that Galerkin discretizations of boundary value problems are convergent.

THEOREM 8.1. *Using the previous notation, let the sequences of matrices* $\{\mathbf{A}_n\}$ *and* $\{\mathbf{B}_n\}$ *be defined via the standard Galerkin discretization of the operators* $\mathcal{A}$ *and* $\mathcal{B}$ *using the sequence of subspaces* $\{V_n\}$. *Then all points in the spectrum of the preconditioned operator*

$$\mathcal{B}^{-1}\mathcal{A} : V \to V$$

are approximated, to an arbitrary accuracy, by the eigenvalues of the preconditioned matrices in the sequence $\{\mathbf{B}_n^{-1}\mathbf{A}_n\}$. *That is, for any point* $\lambda \in \mathrm{sp}(\mathcal{B}^{-1}\mathcal{A})$ *and any* $\epsilon > 0$, *there exists* n^* *such that for all* $n \geq n^*$ *the preconditioned matrix* $\mathbf{B}_n^{-1}\mathbf{A}_n$ *has an eigenvalue* λ_j *satisfying* $|\lambda - \lambda_j| < \epsilon$.

From the point of view of the spectral theory of self-adjoint operators in Hilbert spaces, this theorem expresses the lower semicontinuity of the spectrum of the preconditioned operator $\mathcal{B}^{-1}\mathcal{A}$; cf., e.g., [7, 14, 4]. Since the setting also covers the case when $\mathcal{B}^{-1}\mathcal{A}$ is continuously invertible on the infinite-dimensional Hilbert space V, its finite-dimensional range approximations (that are compact) cannot converge to $\mathcal{B}^{-1}\mathcal{A}$ in norm. Therefore, the result relies upon a pointwise (strong) convergence and the approximations of the individual members of $\mathrm{sp}(\mathcal{B}^{-1}\mathcal{A})$ are not uniform.

From the perspective of the classical approximation theory of weakly formulated PDE eigenvalue problems, the situation warrants the following comments:

- First, here we numerically approximate *the entire spectrum* of $\mathcal{B}^{-1}\mathcal{A}$, which, in general, is not a compact operator. In the numerical PDE literature, however, one typically investigates the point spectrum, i.e., the eigenvalues, using the construction of the so-called *solution operator* that is proven to be compact (the Babuška–Osborn theory; see [2, 3]), or one simply assumes compactness. For a discussion of finite element approximations of noncompact eigenvalue problems with an instructive description of related difficulties, we refer the reader to [23].
- Second, when considering the preconditioned operator $\mathcal{B}^{-1}\mathcal{A}$, there seems to be some ambiguity in describing the relationship among its entire spectrum, the eigenvalue problem $\mathcal{B}^{-1}\mathcal{A}v = \lambda v$, and the generalized eigenvalue problem $\mathcal{A}v = \lambda\mathcal{B}v$. A straightforward discretization of the latter yields the generalized algebraic eigenvalue problem $\mathbf{A_n}\mathbf{v} = \lambda\mathbf{B_n}\mathbf{v}$. In the literature it seems to be considered that the generalized algebraic eigenvalues approximate the generalized PDE eigenvalues. Nevertheless, as Theorem 8.1 shows, within the given setting, the generalized algebraic eigenvalues approximate the entire spectrum of the preconditioned operator. A link to the PDE eigenvalue problem is set aside.[5] Babuška and Osborn avoid this difficulty and present the rigorous definition of the generalized PDE eigenvalues, eigenfunctions, and their approximations through a construction based on compactness; see [2, section 2]. Some of the other literature is not so precise and works with the notion of the generalized PDE eigenvalue problem and its numerical approximation without rigorous specifications.

In relation to the results presented in this paper, it seems that the generalized PDE eigenvalue problem, without assuming compactness, requires further investigations.

9. Open Problems.

In this paper we have rigorously analyzed 2D problems. The original SINUM paper left open the question of whether our main result, Theorem 1.1, also holds in three or even higher dimensions. This has now been positively answered in [21].

Another important issue mentioned in the SINUM paper is the task of "translating" our findings to discretized operators. This was accomplished in [9] for uniformly elliptic operators with scalar coefficient functions. That is, [9] contains discrete versions of the results published in [20] and further progresses towards approximating the individual eigenvalues locally. Extensions to more general preconditioners are presented in [10]. The techniques employed in [9] can be generalized to analyze discretized second order differential operators with indefinite tensors, but such a development is beyond the scope of this paper.

An interesting question concerns the distribution of the eigenvalues of the discretized operators within the interval (1.8). Our numerical experiments suggest that, in the limit $h \to 0$, they cover the whole interval. This is justified by the results in [10], which are briefly presented together with a short discussion on the relationship to the weakly posed PDE eigenvalue problems in section 8. Nevertheless, the approximation of the spectrum of preconditioned PDE operators should be further explored, especially in connection with FEM discretizations, and one can potentially benefit from recent beautiful results regarding smoothed approximations of spectral measures for infinite-dimensional self-adjoint operators [6].

[5] This is true apart from Lemmas 3.2 and 5.2 which prove, provided the tensor K is constant on an open subdomain, that a part of the spectrum of the preconditioned PDE operator is given by a subinterval composed entirely of the generalized PDE eigenvalues.

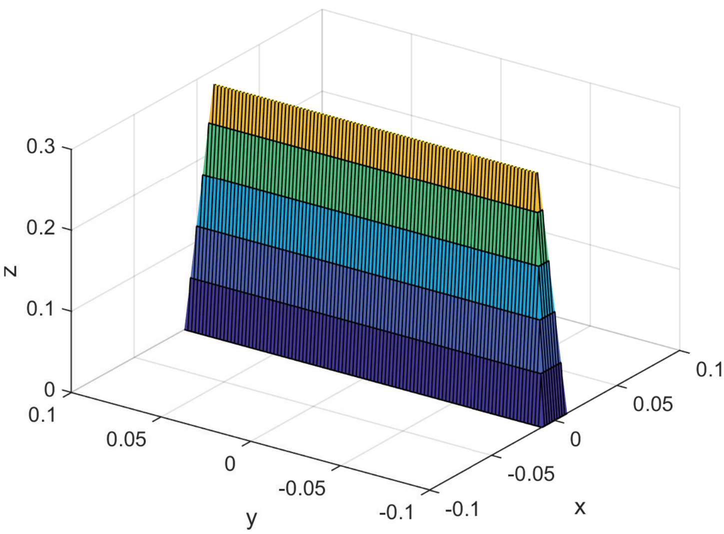

Fig. 6 *The function v_r centered at the point $(0,0)$ with $r = 0.1$.*

Appendix A. Technical Details about Inequalities (3.5) in the Proof of Lemma 3.1. We want to prove that, for sufficiently small $r > 0$,

$$4 - 4r \leq \|(v_r)_x\|^2_{L^2(\Omega)} \leq 4, \tag{A.1}$$

$$\|(v_r)_y\|^2_{L^2(\Omega)} \leq 4r, \tag{A.2}$$

where $v_r(x,y)$ is defined on R_r by (3.3) and (3.4). Without loss of generality, we consider the case $(x_0, y_0) = (0,0)$. Then $R_r = [-r^2, r^2] \times [-r, r]$ and

$$v_r(x,y) = \sqrt{r} \min \left\{1 - \tfrac{|x|}{r^2}, \tfrac{1}{r} - \tfrac{|y|}{r^2}\right\} \quad \text{for } (x,y) \in R_r,$$

with $v_r(x,y) = 0$ elsewhere; see Figure 6.

For any $0 < r < 1$, the partial derivatives of $v_r(x,y)$ are not defined at the boundary ∂R_r of R_r, at the set of points $\{(x,y) \in R_r : |y| - |x| = r - r^2\}$, and at the set of points $\{(x,y) : x = 0,\, |y| < r - r^2\}$ where $v_r(x,y)$ reaches its maximum; see the edges of $\{v_r(R_r)\}$ in Figure 6. Simple computations yield that within R_r,

$$\begin{aligned} |\partial_x v_r(x,y)|^2 &= 0, \quad |\partial_y v_r(x,y)|^2 = \tfrac{1}{r^3} \quad \text{for} \quad |y| - |x| > r - r^2,\ (x,y) \notin \partial R_r, \\ |\partial_x v_r(x,y)|^2 &= \tfrac{1}{r^3}, \quad |\partial_y v_r(x,y)|^2 = 0 \quad \text{for} \quad x \neq 0,\ |y| - |x| < r - r^2,\ (x,y) \notin \partial R_r. \end{aligned}$$

The upper bound in (A.1) thus holds because

$$\|(v_r)_x\|^2_{L^2(\Omega)} = \int_{R_r} |\partial_x v_r(x,y)|^2 \leq \int_{R_r} \tfrac{1}{r^3} = \tfrac{2r^2 \cdot 2r}{r^3} = 4. \tag{A.3}$$

Moreover, denoting

$$P_r = \{(x,y) : x \neq 0,\, |x| < r^2,\, |y| < r - r^2\},$$

we have

$$|\partial_x v_r(x,y)|^2 = \tfrac{1}{r^3}, \quad |\partial_y v_r(x,y)|^2 = 0 \quad \text{for} \quad (x,y) \in P_r.$$

Thus $\|(v_r)_x\|^2_{L^2(\Omega)}$ and $\|(v_r)_y\|^2_{L^2(\Omega)}$ can be bounded as follows:

$$\|(v_r)_x\|^2_{L^2(\Omega)} = \int_{R_r} |\partial_x v_r(x,y)|^2 \geq \int_{P_r} |\partial_x v_r(x,y)|^2 = \int_{P_r} \tfrac{1}{r^3} = \tfrac{2r^2 \cdot 2(r-r^2)}{r^3} = 4 - 4r,$$
$$\|(v_r)_y\|^2_{L^2(\Omega)} = \int_{R_r} |\partial_y v_r(x,y)|^2 = \int_{R_r \setminus P_r} |\partial_y v_r(x,y)|^2 \leq \int_{R_r \setminus P_r} \tfrac{1}{r^3} = \tfrac{2r^2 \cdot 2r^2}{r^3} = 4r,$$

which completes the proof.

Appendix B. Technical Details about the Bound (3.18) in the Proof of Lemma 3.3. Assume that $\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I}$ has a bounded inverse. We will show that, for sufficiently small $l > 0$,

$$\|(\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})^{-1}\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_{\mathcal{L}} \leq \|(\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})^{-1}\|_{\mathcal{L}} \|\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_{\mathcal{L}} < 1. \tag{B.1}$$

The operator norm

$$\|\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_{\mathcal{L}} \equiv \sup_{u \in H_0^1(\Omega)} \frac{\|\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}}$$

can be expressed as (see, e.g., [5, Theorem 4.1–3])

$$\|\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_{\mathcal{L}} = \sup_{u,v \in H_0^1(\Omega)} \frac{\left|\left(\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})u, v\right)_{\mathcal{L}}\right|}{\|u\|_{\mathcal{L}}\|v\|_{\mathcal{L}}}. \tag{B.2}$$

Using

$$\begin{aligned}
\left|\left(\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})u, v\right)_{\mathcal{L}}\right| &= |\langle(\mathcal{A}_l - \mathcal{A})u, v\rangle| \\
&= \left|\int_{S_l} (K(x_0,y_0) - K(x,y))\nabla u \cdot \nabla v\right| \\
&\leq \int_{S_l} \|K(x_0,y_0) - K(x,y)\| |\nabla u| \cdot |\nabla v| \\
&\leq \sup_{(x,y)\in S_l} \|K(x_0,y_0) - K(x,y)\| \, \|u\|_{\mathcal{L}}\|v\|_{\mathcal{L}},
\end{aligned}$$

we get the bound

$$\|\mathcal{L}^{-1}(\mathcal{A}_l - \mathcal{A})\|_{\mathcal{L}} \leq \sup_{(x,y)\in S_l} \|K(x_0,y_0) - K(x,y)\|.$$

Since $\|(\mathcal{L}^{-1}\mathcal{A} - \lambda\mathcal{I})^{-1}\|_{\mathcal{L}}$ is bounded, the continuity of $K(x,y)$ at the point (x_0,y_0) ensures that l can be chosen such that (B.1) holds.

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