Lanczos tridiagonalization, Golub-Kahan bidiagonalization and core problem

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1 Introduction

Consider an orthogonally invariant linear approximation problem $Ax \approx b$. In [8] it is proved that the partial upper bidiagonalization of the extended matrix $[b, A]$ determines a core approximation problem $A_{11}x_1 \approx b_1$, with all necessary and sufficient information for solving the original problem given by $b_1$ and $A_{11}$. It is shown how the core problem can be used in a simple and efficient way for solving different formulations of the original approximation problems. In [3] the core problem formulation is derived from the relationship between the Golub-Kahan bidiagonalization [2] and the Lanczos tridiagonalization [5], and from the known properties of Jacobi matrices. Here we briefly recall the approach from [3], and outline a possible direction for further research.

2 Core problem

Consider estimating $x$ from the real linear approximation problem

$$Ax \approx b, \quad A \text{ a nonzero } n \text{ by } m \text{ matrix}, \quad b \text{ a nonzero } n\text{-vector},$$

where the uninteresting case is excluded by the assumption $A^Tb \neq 0$. In the paper [8] it was proposed to transform the original data $[b, A]$ into the form

$$P^T[ b \parallel AQ ] = \begin{bmatrix} b_1 \\ 0 \\ \alpha \end{bmatrix} \begin{bmatrix} 11 \\ 0 \\ A_{22} \end{bmatrix},$$

where $P^{-1} = P^T$, $Q^{-1} = QT$, $b_1 = \beta v_1$ and $A_{11}$ is a lower bidiagonal matrix with nonzero bidiagonal elements. The matrix $A_{11}$ is either square, when (1) is compatible, or rectangular, when (1) is incompatible. The original problem is in this way decomposed into the approximation problems $A_{11}x_1 \approx b_1$ and $A_{22}x_2 \approx 0$. It is suggested to solve the first problem, set $x_2 \equiv 0$ and substitute $x = Q[x_1^T, 0]^T$ for the solution of (1). The transformation described above has the following remarkable properties, see [3, Theorem 1.1], with the proof given in [8, Theorem 2.2, 3.2, 3.3].

**Theorem 2.1.** Let $A$ be a nonzero $n$ by $m$ real matrix and $b$ a nonzero real $n$-vector, $A^Tb \neq 0$. Then there exists a decomposition (2), where $b_1 = \beta_1 v_1$ and $A_{11}$ is a lower bidiagonal matrix with nonzero bidiagonal elements. Moreover:

1. The matrix $A_{11}$ has full column rank and its singular values are simple.
2. The matrix $A_{11}$ has minimal dimensions over all orthogonal transformations giving the block structure (2).
3. All components of $b_1 = \beta_1 v_1$ in the left singular vector subspaces of $A_{11}$ are nonzero.

The proof of Theorem 2.1 in [8] is based on the singular value decomposition of the matrix $A$. In [3] the relationship between the Golub-Kahan bidiagonalization and the Lanczos tridiagonalization is used for constructing the proof.

3 Lanczos tridiagonalization and core problem properties

Consider the partial lower Golub-Kahan bidiagonalization of $[b, A]$ in the following form. Given the initial vectors $v_0 \equiv 0$, $u_1 \equiv b/\beta_1$, where $\beta_1 \equiv \|b\| \neq 0$ and $\|\cdot\|$ represents the standard Euclidean norm, the algorithm computes for $i = 1, 2, \ldots$

$$\alpha_i v_i = A^T u_i - \beta_i v_{i-1}, \quad \|v_i\| = 1, \quad \beta_{i+1} u_{i+1} = Av_i - \alpha_i u_i, \quad \|u_{i+1}\| = 1$$

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until \( \alpha_i = 0 \) or \( \beta_{i+1} = 0 \), or until \( i = \min\{n,m\} \). Denote \( U_k \equiv (u_1, \ldots, u_k) \), \( V_k \equiv (v_1, \ldots, v_k) \) the matrices with orthonormal columns, \( L_k \) the square lower bidiagonal matrix with the main diagonal \((\alpha_1, \ldots, \alpha_k)\) and the subdiagonal \((\beta_2, \ldots, \beta_k)\) and \( L_{k+1} \equiv (L_k^T, \beta_{k+1}e_k)^T \). In the rest of this contribution we consider (1) incompatible; the compatible case is simpler and can be treated analogously, see [3]. Then (3) must stop with some \( \alpha_{p+1} = 0 \) or \( p = m \) giving

\[
A^T U_p = V_p L_p^T, \quad AV_p = U_{p+1} L_{p+1}.
\]

Consequently, \( U_{p+1}^T[b, AV_p] = [\beta_1 e_1, L_{p+1}] \equiv [b_1 | A_{11}] \) and \( A_{11} x_1 \equiv L_{p+1} x_1 \approx \beta_1 e_1 \equiv b_1 \) is the incompatible core problem. The matrices \( U_{p+1} \) and \( V_p \) represent the first \((p+1)\) and \( p \) columns of the matrices \( P \) and \( Q \), respectively. Given a real symmetric matrix \( B \) and a starting vector \( w_1, \|w_1\| = 1 \), the Lanczos tridiagonalization algorithm can be written in the matrix form

\[
BW_k = W_k T_k + \delta_{k+1} w_{k+1} e_k^T, \quad W_k^T w_{k+1} = 0,
\]

where \( W_k e_1 = w_1, W_k \) has orthonormal columns and \( T_k \) is a symmetric tridiagonal matrix with positive subdiagonal elements, a Jacobi matrix. Defining \( B \equiv A^T A, w_1 \equiv v_1 = A^T b/\|A^T b\| \), it can be shown (see, e.g., [1]) that the Lanczos tridiagonalization (4) produces the matrices \( W_k \equiv V_k, T_k \equiv L_k^T L_k + \delta_{k+1} \equiv \alpha_{k+1} \beta_{k+1} \). Thus, the matrix \( L_{p+1} \) can be linked to the Jacobi matrix

\[
T_p \equiv L_{p+1}^T L_{p+1} = \begin{pmatrix}
\alpha_1^2 + \beta_2^2 & \alpha_2 \beta_2 \\
\alpha_2 \beta_2 & \alpha_2^2 + \beta_3^2 & \ddots \\
\ddots & \ddots & \ddots \\
\alpha_p \beta_p & \alpha_p^2 + \beta_{p+1}^2 & \ddots \\
\end{pmatrix}, \quad L_{p+1} = \begin{pmatrix}
\alpha_1 \\
\beta_2 \\
\alpha_2 \\
\ddots \\
\beta_p \\
\alpha_p \\
\beta_{p+1}
\end{pmatrix}.
\]

This fact is used in [3], together with the properties of Jacobi matrices (see, e.g., [9]), for proving Theorem 2.1.

The presented relationship may be found useful in applications of the core problem formulation. In large ill-posed problems the outer Golub-Kahan bidiagonalization can be combined with an inner regularization applied to the problem \( L_{k+y} \approx \beta_1 e_1 \). Here stopping criteria are typically based on estimation of the L-curve, the discrepancy principle or generalized cross validation (see, e.g., [6], [7]). From the core problem point of view one should ask whether and when the matrix \( L_{k+y} \) for \( k < p \) (possibly \( k \ll p \)) can be considered a sufficiently good approximation to the core matrix \( L_{p+1} \). When \( p \ll m \), one must ask how to numerically indicate the separation of the core problem, since in finite precision computation \( \alpha_{p+1} \) will hardly be identically zero. Similarly, one can ask when the tridiagonal matrix \( T_k, k < p \), sufficiently approximates the matrix \( T_p \) discussed above. It might be useful to study in this context perturbation theory of Jacobi matrices, in particular the specific perturbations when the off-diagonal element \( \delta_{k+1} = \alpha_{k+1} \beta_{k+1} \) is replaced by zero, see [4].

We believe that the presented relationships, together with known results on Jacobi matrices, can be used in further investigation of effective stopping criteria in regularization methods.

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