Behaviour of the smallest singular value while appending a column of a matrix

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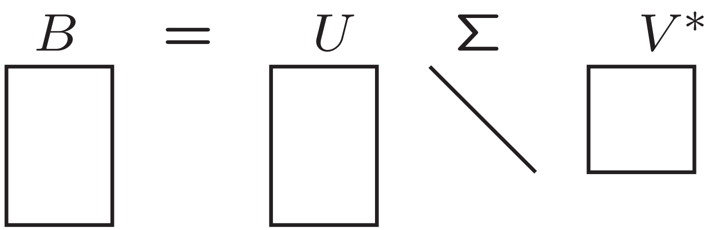
Question

When appending a column to a matrix cause no change of the smallest singular value;
and, if it cause a change, when the change is small?

Solution

Paige, S. Numerische Mathematik (2002) 91: 117–146,
Paige, S. Numerische Mathematik (2002) 91: 91–115,
Wilkinson, Algebraic Eigenvalue Problem, (1965) §39,
Golub, Bunch, Nielsen, Elsner, Mehrmann, He, . . .

B an n by k matrix, c an n-vector, $n > k$, rank$(B) = k$, $\gamma > 0$



SVD of B,

$\delta(\gamma) = \sigma_{\text{min}}([c\gamma, B]) \,/\, \sigma_{\text{min}}(B)$. When $\delta(\gamma) = 1$ $(\delta(\gamma) \to 1)$?

$$\left[\begin{array}{c|c} \gamma^2\|c\|^2 & \gamma(B^*c)^T \\ \hline \gamma B^*c & B^*B \end{array}\right] \approx \left[\begin{array}{c|c} \gamma^2\|c\|^2 & \gamma(\Sigma U^*c)^T \\ \hline \gamma\Sigma U^*c & \Sigma^2 \end{array}\right]$$

A zero entry in U^*c indicates trivial preserving of the corresponding singular value.

Denote by

$$\sigma_1(B) > \cdots > \sigma_{t-1}(B) > \sigma_t(B) \equiv \sigma_{\min}(B)$$

the distinct singular values of B with multiplicities $r_1, \ldots, r_t$. Then the s. v. of $[c\gamma, B]$ must solve the characteristic equation

$$(\gamma^2 \|c\|^2 - \sigma^2) \prod_{i=1}^{t} (\sigma_i^2 - \sigma^2)^{r_i}$$
$$- \gamma^2 \sum_{i=1}^{t} \sigma_i^2 \|c_i\|^2 (\sigma_i^2 - \sigma^2)^{r_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{t} (\sigma_j^2 - \sigma^2)^{r_j} = 0$$

where $\|c_i\|$ denotes the size of the component of the vector c in the subspace generated by the left singular vectors of B corresponding to σ_i.

Each singular value of B with $r_i > 1$ must be at least $r_i - 1$ multiple singular value of $[c\gamma, B]$. Singular values of $[c\gamma, B]$ different from the singular values of B must solve the deflated secular equation

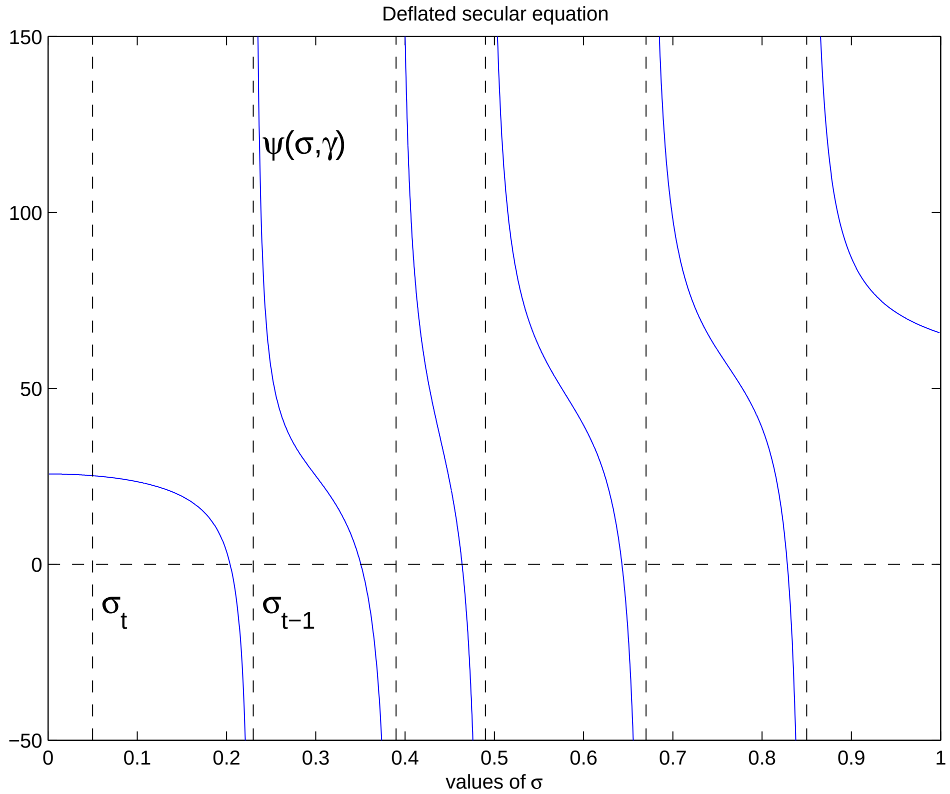
$$\gamma^2 \|c\|^2 - \sigma^2 - \gamma^2 \sum_{i=1}^{t} \|c_i\|^2 \frac{\sigma_i^2}{\sigma_i^2 - \sigma^2} = 0 .$$

Denoting by r the residual of the LS problem for $Bx \approx c$,

$$\|r\|^2 = \|c\|^2 - \sum_{i=1}^{t} \|c_i\|^2 ,$$

the deflated secular equation gets the form

$$\psi(\sigma,\gamma) \equiv \gamma^2 \|r\|^2 - \sigma^2 - \gamma^2\sigma^2 \sum_{i=1}^{t} \frac{\|c_i\|^2}{\sigma_i^2 - \sigma^2} = 0\,.$$



Recall the notation

$$\sigma_1(B) > \cdots > \sigma_{t-1}(B) > \sigma_t(B) \equiv \sigma_{\min}(B)$$

Theorem 1: Necessary and sufficient condition

$$\sigma_{\min}([c\gamma, B]) = \sigma_{\min}(B) \qquad (\delta(\gamma) = 1)$$

if and only if

$$\|c_t\| = 0 \qquad \text{and} \qquad \psi(\sigma_{\min}(B), \gamma) \geq 0.$$

Assume $c_t \neq 0$. Then

$$\sigma_{\min}([c\gamma, B]) \equiv \sigma_{t+1}([c\gamma, B]) \; < \; \sigma_{\min}(B), \qquad \delta(\gamma) < 1 .$$

Theorem 2: Bounds

$$\delta(\gamma) \leq \frac{\|r\|}{\left(\|c_t\|^2 + \|r\|^2\right)^{1/2}} < 1, \qquad \delta(\gamma) \leq \frac{\gamma\|r\|}{\sigma_t([c\gamma, B])} .$$

Theorem 3: Dependence on γ in $([c\gamma, B])$

For $\quad \gamma < \sigma_{\min}(B)/\|c\| \quad$ we get $\quad \delta(\gamma) < 1.$

For $\quad \gamma = \widehat{\gamma} \equiv \sigma_{\min}(B)/\|c\|$

$$\delta(\widehat{\gamma}) = 1 \quad \text{if and only if} \quad \|r\| = \|c\| \quad (\|c_i\| = 0,\ i = 1, \ldots, t).$$

Moreover, with $\widehat{\gamma}$ we have the following bound for the LS residual

$$\delta(\widehat{\gamma})\,\|c\| \;\leq\; \|r\| \;\leq\; \delta(\widehat{\gamma})\,\|c\|\,\left\{2 - \delta(\widehat{\gamma})^2\right\}^{1/2}.$$

Liesen, Rozložník, S, SISC (2002), 23: 1503–1525

What is it good for?

I. Basic solution of the (Scaled) Total Least Squares

II. Fundamentals of (Scaled) Total Least Squares:

- A consistent theory ($c_t = 0 \Rightarrow \sigma_{\mathrm{min}}(B)$ should play no role!)
- A new approach to single right hand side problems based on the bidiagonalization of $[c, B]$

III. Rounding error analysis of MGS GMRES:
$\delta \rightarrow 1$ disturbes the bounds. However, small residual norm cause $\delta \ll 1$.

Sherman2, b MM, $x_0 = 0$