Infinite combinatorics in homological algebra

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Large-Cardinal Methods in Homotopy
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Outline

1. Cotorsion pairs
2. Small object argument and related
3. Deconstruction
4. Hunter's cardinal argument

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Definition of a cotorsion pair

Notation: R a ring, Mod$R$ the category of R-right modules.

Definition (Salce, 1977)

Let $\mathcal{X}, \mathcal{Y}$ be two classes of modules. The pair $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair if

$$\mathcal{X} = {}^{\perp}\mathcal{Y} \stackrel{def.}{=} \{X \mid \mathrm{Ext}^1_R(X, Y) = 0 \ \forall Y \in \mathcal{Y}\}$$
$$\mathcal{Y} = \mathcal{X}^{\perp} \stackrel{def.}{=} \{Y \mid \mathrm{Ext}^1_R(X, Y) = 0 \ \forall X \in \mathcal{X}\}$$

The cotorsion pair is complete if for each $M \in \mathrm{Mod}R$, there are short exact sequences

$$0 \to M \to Y \to X \to 0 \quad \text{and} \quad 0 \to Y' \to X' \to M \to 0$$

such that $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$. beyond

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Beyond module categories

- Cotorsion can be defined in more general additive categories $\mathcal{C}$.
- We need a class $\mathcal{E}$ of diagrams of the form

$$0 \longrightarrow Y \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

  playing the role of short exact sequences and some suitable axioms for these.
- Such a pair $(\mathcal{C}, \mathcal{E})$ is called an exact category [Quillen 1972; Keller 1990].
- The condition $\mathrm{Ext}^1(X, Y) = 0$ means that each designated sequence above splits. That is, there exist morphisms r and s such that

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- Once we give a meaning of a short exact sequence and the vanishing of Ext^1 in $\mathcal{C}$, we can define a complete cotorsion pair there. ▸ definition
- Various problems about localization of algebraic triangulated categories formally translate to problems about cotorsion pairs in exact categories.
- Parts of the theory for modules has been generalized [Saorín-Š. 2011].

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Are there incomplete cotorsion pairs?

- Shortly, I will discuss techniques for proving that a given cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is complete.
- Paradox: It seems much harder to prove that a cotorsion pair is not complete.
- Facts: The cotorsion pair $({}^{\perp}\mathbb{Z}, ({}^{\perp}\mathbb{Z})^{\perp})$ in Ab is not complete in certain consistent extension of ZFC [Eklof-Shelah 2003]. But it is complete in another consistent extension of ZFC (e.g. V=L).
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Filtrations

Definition

Let $X \in \text{Mod}R$. A filtration of X is a well ordered chain

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \mathcal{X}_\alpha \subseteq X_{\alpha+1} \subseteq \cdots \subseteq X_\sigma = X$$

of submodules of X such that for all limit ordinals $\alpha \leq \sigma$:

$$X_\alpha = \bigcup_{\beta<\alpha} X_\beta \qquad \text{(no gaps!)}$$

Suppose that $\mathcal{S} \subseteq \text{Mod}R$ is a class of modules. We call X an $\mathcal{S}$-filtered module (or a transfinite extension of modules from $\mathcal{S}$) if there is a filtration $(X_\alpha \mid \alpha \leq \sigma)$ such that up to isomorphism

$$X_{\alpha+1}/X_\alpha \in \mathcal{S} \quad \text{for each } \alpha + 1 \leq \sigma.$$

Denote by $\text{Filt}\,\mathcal{S}$ the class of all $\mathcal{S}$-filtered modules.

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Closure properties and the small object argument

- Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in Mod$R$.
- Clearly, $R \in \mathcal{X}$ (since R is projective) and $\mathcal{X}$ is closed under retracts.

Lemma (Auslander; Eklof, 1977)

$\mathcal{X}$ is closed under transfinite extensions. That is, an $\mathcal{X}$-filtered module belongs to $\mathcal{X}$.

Theorem (Eklof-Trlifaj, 2001; the idea is older: Quillen 1967)

Let $\mathcal{S}$ be a set of R-modules containing R. Then the cotorsion pair

$$(\mathcal{X}, \mathcal{Y}) \stackrel{def.}{=} ({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$$

is complete. Moreover, for each module X we have:

$$X \in \mathcal{X} \iff X \text{ is a retract of an } \mathcal{S}\text{-filtered module.}$$

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- We can do better.
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Outline

1 Cotorsion pairs

2 Small object argument and related

3 Deconstruction

4 Hunter's cardinal argument

Deconstruction

- To prove that we have a complete cotorsion pair, it is often enough to prove that some class $\mathcal{X}$ is of the form $\operatorname{Filt}\mathcal{S}'$, $\mathcal{S}'$ a set.
- This is called deconstruction—we want to "deconstruct" each $X \in \mathcal{X}$ to a filtration with "small composition factors", again in $\mathcal{X}$.
- The usual procedure:
  1. Choose a cardinal μ, a bound for the size of composition factors.
  2. Prove that each module $X \in \mathcal{X}$ of cardinality $\kappa > \mu$ admits a filtration
  $$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \mathcal{X}_\alpha \subseteq X_{\alpha+1} \subseteq \cdots \subseteq X_\sigma = X$$
  such that each $X_{\alpha+1}/X_\alpha$ belongs to $\mathcal{X}$ and has cardinality $< \kappa$.
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- Methods to achieve Step 2 above are typically very different when κ is regular compared to when κ is singular.

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- Suppose $X \in \mathcal{X}$ and $\kappa = |X|$ is a regular cardinal greater than or equal to μ. We typically proceed in two steps:
  1. We find a filtration $(X_\alpha \mid \alpha \leq \sigma)$ such that $|X_\alpha| < \kappa$ and $X_\alpha \in \mathcal{X}$ for all $\alpha < \sigma$. So far it may well happen that $X_{\alpha+1}/X_\alpha \notin \mathcal{X}$!
  2. We hope to find a closed unbounded subset $C \subseteq \kappa$ such that $(X_\alpha \mid \alpha \in C)$ has all consecutive factors in $\mathcal{X}$.
- Step 1 is specific to the class $\mathcal{X}$. For example if $R = \mathbb{Z}$, the left hand class of every cotorsion pair $\mathcal{X}$ is closed under submodules and the filtration comes for free.
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- Suppose $X \in \mathcal{X}$ and $\kappa = |X|$ is a regular cardinal greater than or equal to μ. We typically proceed in two steps:
  1. We find a filtration $(X_\alpha \mid \alpha \leq \kappa)$ such that $|X_\alpha| < \kappa$ and $X_\alpha \in \mathcal{X}$ for all $\alpha < \kappa$. So far it may well happen that $X_{\alpha+1}/X_\alpha \notin \mathcal{X}$!
  2. We hope to find a closed unbounded subset $C \subseteq \kappa$ such that $(X_\alpha \mid \alpha \in C)$ has all consecutive factors in $\mathcal{X}$.
- Step 1 is specific to the class $\mathcal{X}$. For example if $R = \mathbb{Z}$, the left hand class of every cotorsion pair $\mathcal{X}$ is closed under submodules and the filtration comes for free.
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Stationary sets

Definition

Let κ be an uncountable regular cardinal. We say that two subsets $S, T \subseteq \kappa$ are equivalent, $S \sim T$, if there exist a closed unbounded subset $C \subseteq \kappa$ such that

$$S \cap C = T \cap C.$$

A subset $S \subseteq \kappa$ is called stationary if $[S]_\sim \neq [\emptyset]_\sim$. In other words, S intersects every closed unbounded subset of κ.

Example

Let $\lambda < \kappa$ be another regular cardinal. Then

$$S_\lambda = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \lambda\}$$

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- Recall: So far we have a filtration $(X_\alpha \mid \alpha \le \kappa)$ such that $|X_\alpha| < \kappa$ and $X_\alpha \in \mathcal{X}$ for all $\alpha < \kappa$.
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The equivalence class $[E]_\sim$ does not depend on the choice of the filtration $(X_\alpha \mid \alpha \le \kappa)$.

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X admits a filtration $(X'_\alpha \mid \alpha \le \kappa)$ with $|X'_\alpha| < \kappa$ and $X'_{\alpha+1}/X'_\alpha \in \mathcal{X}$ for all $\alpha < \kappa$ if and only if $\Gamma(X) = [\emptyset]_\sim$ if and only if E is not stationary.

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▸ illustration

Shelah's Singular Compactness for groups

- What to do for singular cardinals? The methods for regular cardinals do not work, but we have:

Theorem (Shelah, 1974)

If κ is a singular cardinal and X is an abelian group of cardinality κ, all of whose subgroups of strictly smaller cardinality are free, then X is free.

- Consequence: Assuming $\diamondsuit_\kappa$ for each regular κ (follows from V=L, consistent with ZFC) then an abelian group belongs to ${}^\perp\mathbb{Z}$ if and only if it is free.
- Some other combinatorial principles may prevent deconstruction at regular cardinals: It is consistent with ZFC that there are non-free groups in ${}^\perp\mathbb{Z}$.
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Theorem (Eklof-Fuchs-Shelah, 1990)

Let $\mathcal{S}$ be a set of modules and μ be a cardinal such that each $S \in \mathcal{S}$ is $\leq \mu$-presented. Suppose we are given a singular cardinal $\kappa > \mu$, a κ-generated module X, and for each regular cardinal λ such that $\mu < \lambda < \kappa$ a set $\mathcal{C}_\lambda$ of λ-generated submodules of X satisfying:

1. *every element of $\mathcal{C}_\lambda$ is $\mathcal{S}$-filtered;*
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Then X is $\mathcal{S}$-filtered.

1. If R is fixed and $\kappa \gg 0$, then:
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Shelah's Singular Compactness more generally

Theorem (Eklof-Fuchs-Shelah, 1990)

Let $\mathcal{S}$ be a set of modules and μ be a cardinal such that each $S \in \mathcal{S}$ is $\leq \mu$-presented. Suppose we are given a singular cardinal $\kappa > \mu$, a κ-generated module X, and for each regular cardinal λ such that $\mu < \lambda < \kappa$ a set $\mathcal{C}_\lambda$ of λ-generated submodules of X satisfying:

1. *every element of $\mathcal{C}_\lambda$ is $\mathcal{S}$-filtered;*
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Applications of the deconstruction methods

1. The solution to the Whitehead problem: It is independent of ZFC whether all abelian groups in ${}^{\perp}\mathbb{Z}$ are free. [Shelah 1974]
2. The solution to the Baer splitting problem: Over a commutative domain R, a module X is projective if and only if $\mathrm{Ext}^1_R(X, T) = 0$ for any torsion module T. [Eklof-Fuchs-Salce 1990], [Angeleri-Bazzoni-Herbera 2008]
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  1. the Γ-invariant,
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Outline

1. Cotorsion pairs
2. Small object argument and related
3. Deconstruction
4. Hunter's cardinal argument

Hunter's lemma

- There is another way to prove $\mathrm{Ext}^1(X, Y) = 0$ using set theory:

Lemma (Hunter, 1976)

Let X, Y be modules and suppose we have an exact sequence

$$\varepsilon: \quad 0 \longrightarrow P \longrightarrow E \longrightarrow X^{(I)} \longrightarrow 0,$$

$|\mathrm{Hom}_R(P, Y)| < 2^{|I|}$ *and* $\mathrm{Ext}^1_R(E, Y) = 0$*. Then* $\mathrm{Ext}^1_R(X, Y) = 0$*.*

Proof

- Applying $\mathrm{Hom}_R(-, Y)$ to ε, we get an exact sequence $\mathrm{Hom}_R(P, Y) \to \mathrm{Ext}^1_R(X^{(I)}, Y) \to \mathrm{Ext}^1_R(E, Y) = 0$.
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Infinite combinatorics in homological algebra

Jan Šťovíček

Charles University in Prague

Large-Cardinal Methods in Homotopy
September 3^{rd}, 2011

Rich systems of submodules from sums

- Let μ be a regular cardinal and $\mathcal{S}$ be a set of $< \mu$-presented modules not containing 0.
- Suppose we have a module of the form $M = \bigoplus_{\alpha<\sigma} S_\alpha$.
- Then we can find a distributive complete sublattice $\mathcal{L}$ of submodules of M such that :
    1. $0 \in \mathcal{L}$ and $M \in \mathcal{L}$,
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The regular case—an illustration

- Methods for proving that $\Gamma(X) = [\emptyset]_\sim$: either specific to the situation or with the aid of combinatorial principles.
- Fix a module Y and put $\mathcal{X} = {}^\perp Y$.
- Suppose we have $X \in \mathcal{X}$ of cardinality κ for $\kappa \geq |Y|$ regular and we have succeeded in finding a filtration $(X_\alpha \mid \alpha \leq \kappa)$ such that $|X_\alpha| < \kappa$ and $X_\alpha \in \mathcal{X}$ for all $\alpha < \kappa$.
- We can be lucky and have Jensen's Diamond Principle $\diamondsuit_\kappa$ at our disposal (a combinatorial principle which is independent of ZFC):

Definition ($\diamondsuit_\kappa$)

For every stationary set E, there is a set of functions $f_\alpha : X_\alpha \to Y \times X_\alpha$ ($\alpha \in E$) such that for any function $f : X \to Y \times X$, the set

$$\{\alpha \in E \mid f_\alpha = f|_{X_\alpha}\}$$

remains stationary (in particular it is non-empty).

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$$E = \{\alpha < \kappa \mid (\exists \beta)(\alpha < \beta < \kappa \text{ and } X_\beta / X_\alpha \notin \mathcal{X})\}$$

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- This allows us to take $E' \sim E$ and construct a filtration

$$0 \subseteq Y = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\alpha \subseteq F_\beta \subseteq \cdots \subseteq F_\kappa = F$$

such that for each $\alpha < \kappa$ we have a split exact sequence

$$\varepsilon_\alpha : \quad 0 \longrightarrow Y \xrightarrow{\subseteq} F_\alpha \longrightarrow X_\alpha \longrightarrow 0,$$

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