# A counterexample to Rosický's problem

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# Outline

1. The problem and motivation
2. The case of discrete valuation domains
   - Balanced sequences and Walker's modules
   - Employing the $p^\lambda$-adic topology
3. The counterexample
   - Purity in finitely accessible categories
   - Results of Osofsky and Lenzing
   - Summary

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# The problem and motivation

## Problem (Rosický)

*Given a ring $R$, is there a regular cardinal $\lambda$ such that the $\lambda$-pure global dimension of Mod-$R$ is $\leq 1$?*

## Motivation

Representability of functors in triangulated categories. In this case: Obstructions to representability of certain functors $\mathbf{D}(\text{Mod-}R) \to \mathbf{Ab}$.

## Theorem (Bazzoni-Š., 2010)

*Let $k$ be an uncountable field (e.g. $k = \mathbb{C}$). Assume that $R$ is one of:*

1. $R = k[x_1, x_2, \ldots, x_n]$, $n \geq 2$,
2. $R = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, $\dim_k V \geq 2$.

*Then no such regular cardinal $\lambda$ exists.*

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  - $p^0 G = G$,
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- Note:

$$p^0 G \supseteq p^1 G \subseteq p^2 G \supseteq \cdots \supseteq p^\sigma G \supseteq p^{\sigma+1} G \supseteq \cdots$$

  is a transfinite sequence of iterated Jacobson radicals.
- The length of $G$ is defined as $\min\{\lambda \mid p^\lambda G = p^{\lambda+1} G\}$. For such $\lambda$, $p^\lambda G$ is divisible, so a summand of $G$. In particular, $p^\lambda G = 0$ if $G$ is reduced.

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# Balanced sequences

## Observation

$G \mapsto p^\lambda G$ gives a functor $p^\lambda(-) : \text{Mod-}R \to \text{Mod-}R$, which is not exact.

## Definition

Let $\lambda$ be an ordinal. A short exact sequence

$$0 \to A \to B \to C \to 0$$

is $\lambda$-balanced if

$$0 \to p^\sigma A \to p^\sigma B \to p^\sigma C \to 0$$

is exact for each $\sigma < \lambda$.

## Aim

Let $\lambda$ be limit. Construct a set of modules $\mathcal{S}_\lambda$ such that

$$\lambda\text{-balanced} \iff \mathcal{S}_\lambda\text{-pure}.$$

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$$\lambda\text{-balanced} \quad \Longleftrightarrow \quad \mathcal{S}_\lambda\text{-pure}.$$

# Walker's modules $P_\beta$

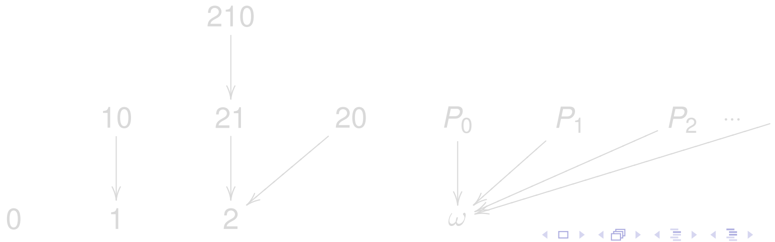
- Construct a module $P_\beta$ using generators and relations.
- For an ordinal $\beta$, generators are indexed by finite sequences

$$\beta\beta_1\beta_2\dots\beta_n \quad \text{such that} \quad \beta > \beta_1 > \beta_2 > \cdots > \beta_n.$$

- Relations:

$$p \cdot \beta_1\beta_2\dots\beta_n\beta_{n+1} = \beta_1\beta_2\dots\beta_n \quad \text{and} \quad p \cdot \beta = 0.$$

- Note: $\beta$ infinite $\implies$ $P_\beta$ is $|\beta|$-presented.



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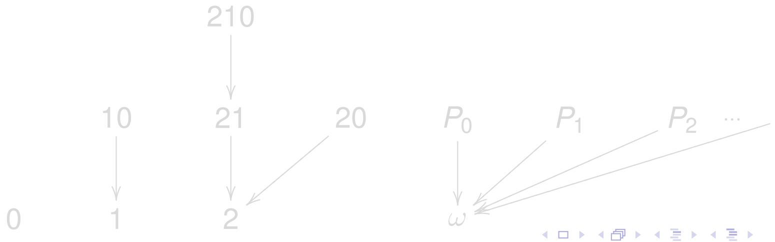

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$$\begin{tikzcd}
 & & 210 \arrow[d] & & & & & \\
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0 & 1 & 2 & & \omega & & &
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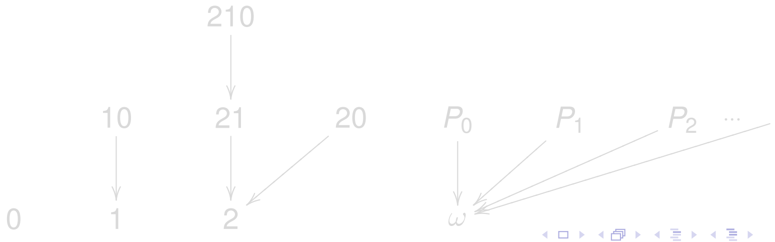

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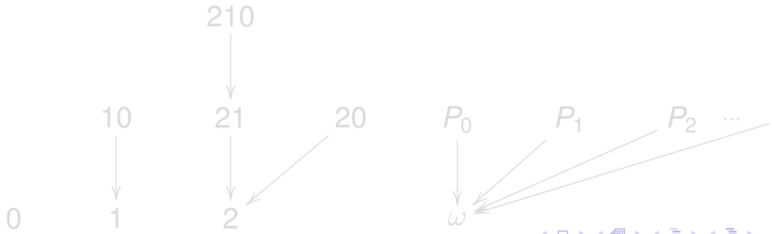

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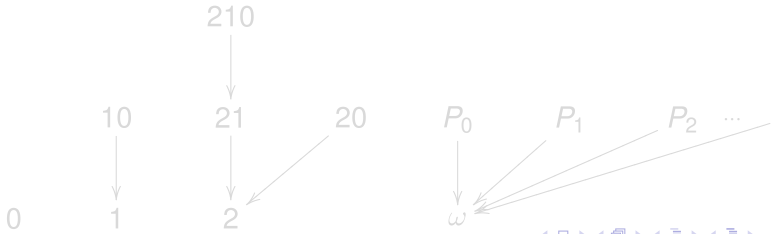

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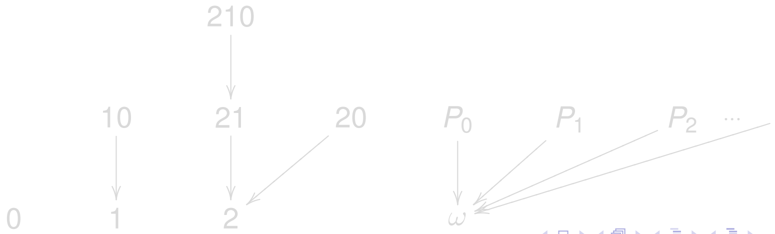

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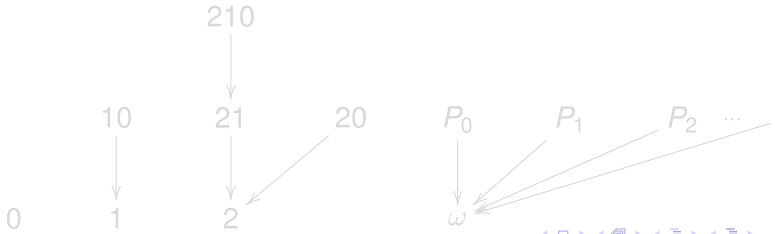

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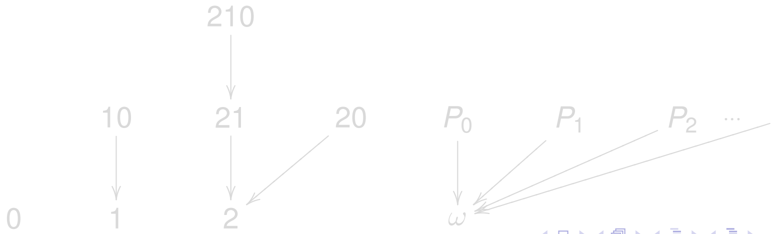

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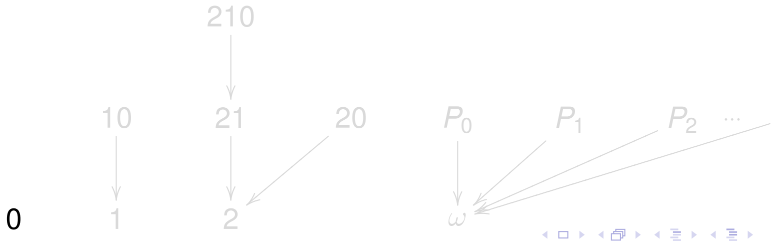

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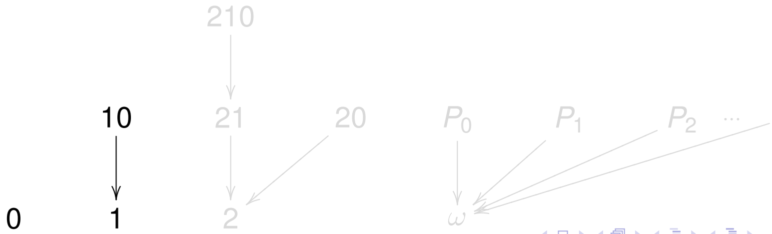

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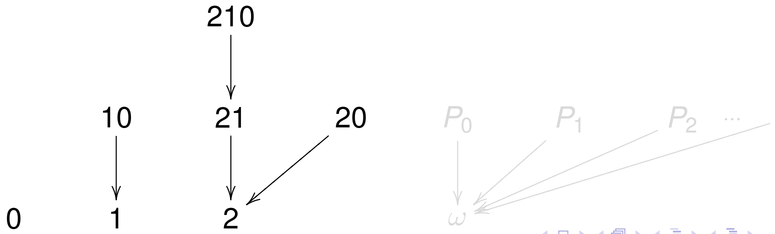

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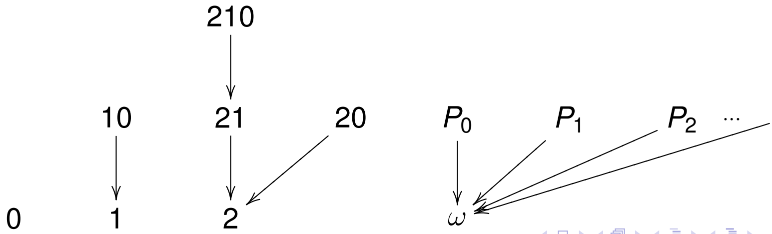

# A characterization of balanced sequences

## Theorem (Walker, 1972)

*Let $\lambda$ be a limit ordinal. The following are equivalent for an exact sequence $\varepsilon : 0 \to A \to B \to C \to 0$:*

1. *$\varepsilon$ is $\lambda$-balanced.*
2. *For each $\sigma < \lambda$,*
$$0 \to \mathrm{Hom}_R(P_\sigma, A) \to \mathrm{Hom}_R(P_\sigma, B) \to \mathrm{Hom}_R(P_\sigma, C) \to 0$$
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- Given a module $G$ and an ordinal $\lambda$, the $p^\lambda$-adic topology on $G$ is a linear topology with basis of neighborhoods of $0 \in G$ taken as

$$\mathcal{U}_0 = \{p^\sigma G \mid \sigma < \lambda\}$$

- For abelian $p$-groups studied by Mines, 1968.

**Facts**

Assume $\lambda$ is limit and $G$ is reduced torsion. Then:

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# Pure projectives are complete for uncountable $\lambda$

## Fact & Definition

Any linear topology determines a uniform space.

So we say that $G$ is complete in the $p^\lambda$-adic topology provided that every Cauchy net converges.

## Theorem (Salce, 1980)

*Let*

- *$R$ be a discrete valuation domain,*
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## Theorem (Bazzoni-Š., 2010)

*Let $R$ be a discrete valuation domain and $\lambda$ an uncountable regular cardinal.*
*Then the $\lambda$-pure global dimension of Walker's module $P_\lambda$ is $> 1$.*

## Idea behind proof

1. The exact sequence
$$0 \longrightarrow K \longrightarrow \bigoplus_{\beta<\lambda} P_\beta^{(\mathrm{Hom}_R(P_\beta,P_\lambda))} \longrightarrow P_\lambda \longrightarrow 0$$
is $\lambda$-pure and $\lambda$-balanced.
2. $K$ is not a closed subspace of $\bigoplus_{\beta<\lambda} P_\beta^{(\mathrm{Hom}_R(P_\beta,P_\lambda))}$ in $p^\lambda$-adic topology, So it is nether complete nor $\lambda$-pure projective.

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*Then the $\lambda$-pure global dimension of Walker's module $P_\lambda$ is $> 1$.*

## Idea behind proof

1. The exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{\beta<\lambda} P_\beta^{(\mathrm{Hom}_R(P_\beta,P_\lambda))} \longrightarrow P_\lambda \longrightarrow 0$$

is $\lambda$-pure and $\lambda$-balanced.

2. $K$ is not a closed subspace of $\bigoplus_{\beta<\lambda} P_\beta^{(\mathrm{Hom}_R(P_\beta,P_\lambda))}$ in $p^\lambda$-adic topology, So it is nether complete nor $\lambda$-pure projective.

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# Outline

1. The problem and motivation
2. The case of discrete valuation domains
   - Balanced sequences and Walker's modules
   - Employing the $p^\lambda$-adic topology
3. The counterexample
   - Purity in finitely accessible categories
   - Results of Osofsky and Lenzing
   - Summary

# Finitely accessible categories

## Definition

Let $\mathcal{C}$ be an category with direct limits. Then

- $X \in \mathcal{C}$ is finitely presentable if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with direct limits.
- $\mathcal{C}$ is a finitely accessible category if $\exists$ set $\mathcal{S}$ of finitely presentable objects such that $\mathcal{C} = \varinjlim \mathcal{S}$.
- $\mathcal{T} \subseteq \mathcal{C}$ is a finitely accessible subcategory of $\mathcal{C}$ if $\mathcal{T}$ is closed under $\varinjlim$ in $\mathcal{C}$ and for each $X \in \mathcal{T}$ we have

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# Purity in finitely accessible categories

## Fact

If $\mathcal{C}$ is additive finitely accessible and $\lambda$ regular, it makes perfect sense to speak of

1. $\lambda$-pure exact sequences and $\lambda$-pure projective objects in $\mathcal{C}$,
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3. $\lambda$-pure global dimension of $\mathcal{C}$.

## Observation (irrelevance of the ambient category!)

If $\mathcal{T} \subseteq \mathcal{C}$ is a finitely accessible subcategory and $G \in \mathcal{T}$, then

$$\lambda\text{-pure proj.dim}_{\mathcal{T}} G = \lambda\text{-pure proj.dim}_{\mathcal{C}} G$$

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# Transferring lower bounds for $\lambda$-pure global dimension

- Let $R$ be a discrete valuation domain and $\lambda$ regular uncountable.
- Let $\mathcal{T}$ be the category of torsion $R$-modules.
- Then $\mathcal{T}$ is a finitely accessible subcategory of Mod-$R$.
- We know that Walker's $P_\lambda$ belong to $\mathcal{T}$.
- Therefore, $\lambda$-pure proj.dim$_\mathcal{T} P_\lambda > 1$.
- If we can embed $\mathcal{T}$ as a finitely accessible subcategory into Mod-$S$ ($S$ another ring). Then

$$1 < \lambda\text{-pure gl.dim } \mathcal{T} \leq \lambda\text{-pure gl.dim (Mod-}S).$$

- We can do this for $S = k[x_1, x_2, \dots, x_n]$, $n \geq 2$, and for $S = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, $\dim_k V \geq 2$!

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# Results of Osofsky and Lenzing

**Theorem (Osofsky, 1973)**

*Let $k$ be an uncountable field and $R = k[x, y]$. Then:*

$$\text{pure proj.dim}_R k(x, y) = 2.$$

**Theorem (Lenzing, 1984)**

*Let $k$ be an uncountable field and $R = \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$. Let $G \in \text{Mod-}R$ be the generic module (analog of the fraction field). Then:*

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# The counterexample

When putting all the pieces together, we obtain:

**Theorem (Bazzoni-Š., 2010)**

*Let $k$ be an uncountable field and $\lambda$ any infinite regular cardinal. Assume that $R$ is one of:*

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