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# Axioms, algorithms and Hilbert's Entscheidungsproblem

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# Outline

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The Decision Problem

Formal Languages and Theories

Incompleteness

Undecidability

# The Decision Problem

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*From: David Hilbert and Wilhelm Ackermann,
Foundations of Theoretical Logic (Grundzüge der theoretischen Logik), 1928.*

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Hilbert & Ackermann:

> *We want to make it clear that for the solution of the decision problem a process would be given . . . , even though the difficulties of the process would make practical use illusory . . .*

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1. The important results on incompleteness and undecidability come from 1930's — well before the first real computers were constructed!
2. Despite the problems, there are computer programs designed for automated proving.

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— Relation symbol: $=$

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(Goldbach's Conjecture).

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(using substitution, modus ponens, generalization, logical axioms).

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2. Complete. If $\xi$ is a sentence, there should be a proof for either $\xi$ or “not $\xi$”.

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**Theorem (Kurt Gödel, 1931)**

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Effective axiomatic description: There is an algorithm which determines whether a given formula is an axiom.

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Effective axiomatic description: There is an algorithm which determines whether a given formula is an axiom.
Recall: $(P7)$ $(\varphi(0)$ and $(\forall x)(\varphi(x) \Longrightarrow \varphi(x+1))) \implies (\forall x)(\varphi(x))$.

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— There can be no proof for $\xi$ in Peano Arithmetics, but $\xi$ is a true statement about natural numbers.

— The second part is newer with a different proof, it can be essentially found in a nice book by Alfred Tarski.

# Summary

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Any axiomatic description of the arithmetics of natural numbers which is

1. consistent, and
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is necessarily incomplete.

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# The Decision Problem Revisited

**Problem (Hilbert's Entscheidungsproblem)**

*Is there an algorithm which, given an effectively described theory, such as Peano Arithmetics, and a sentence $\xi$ in the theory decides, whether $\xi$ is or is not provable from the axioms.*

# The Decision Problem Revisited

**Problem (Hilbert's Entscheidungsproblem)**

*Is there an algorithm which, given an effectively described theory, such as Peano Arithmetics, and a sentence $\xi$ in the theory decides, whether $\xi$ is or is not provable from the axioms.*

In other words:

— We have given up finding a complete axiomatic description for natural numbers.

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# The Decision Problem Revisited

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In other words:

— We have given up finding a complete axiomatic description for natural numbers.

— However, we still want an algorithm for automated proving for the descriptions we have at our disposal.

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2. A Turing machine (after Alan Turing).
3. Lambda calculus (by Alonzo Church).

# What Precisely is an Algorithm?

If we want to prove anything about existence of algorithms we need to have a definition of an algorithm.

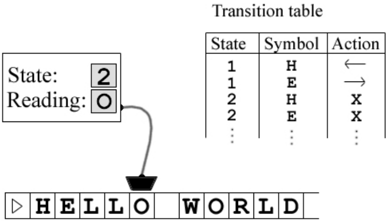
Different computational models:

1. A program in C, Java, Pascal or similar which gets an input file and can write its output to an output file.
2. A Turing machine (after Alan Turing).
3. Lambda calculus (by Alonzo Church).

All the models above have the same computational strength.

# Church-Turing Thesis

A Turing machine:



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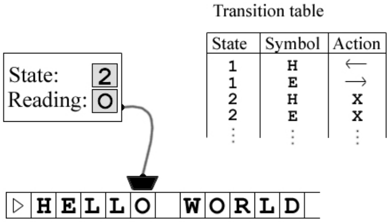


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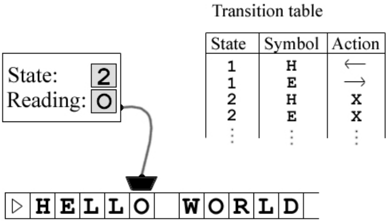


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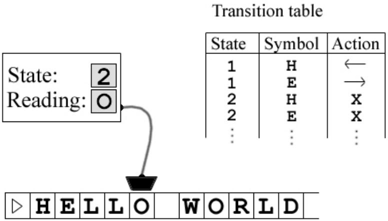


## Church-Turing Thesis

The intuitive notion of an “algorithm” is, formally, a Turing machine which finishes its computation in finite time given any input (= halts on each input).

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**Theorem (Alan Turing, 1936)**

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**Theorem (Alan Turing, 1936)**

*There is no such algorithm. Therefore, the halting problem is undecidable.*

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— What does then $\mathrm{Halt}(\mathrm{Diag}, \mathrm{Diag})$ return?

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*The same holds for any consistent axiomatic description of the artihmetics of natural numbers.*

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- The second part can be found in the book by Tarski.

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