Idempotent ideals in a module category and the Telescope conjecture

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Notation

For an ideal $\mathfrak{I}$, let $\mathfrak{I}(X, Y) = \{f : X \to Y \mid f \in \mathfrak{I}\}$.

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An ideal $\mathfrak{I}$ is idempotent if $\mathfrak{I}^2 = \mathfrak{I}$. Equivalently:

$$(\forall f \in \mathfrak{I})(\exists g, h \in \mathfrak{I}) \quad f = gh$$

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- Often not, but it is true for some non-trivial cases.

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- Notation: $\mathrm{rad}_{\mathcal{C}}$

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Definition "from inside": The ideal of all morphisms f such that there exist an inverse system

$$(f_{pr} : X_r \to X_p \mid p, r \in \mathbb{Q} \cap [0,1], p \leq r)$$

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- By assumption: $h_\alpha = 0$ for $\alpha \gg 0$.

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Condition $(*)$ on $\mathcal{C}$

$\mathcal{C}$ has local d.c.c. on ideals—whenever

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and $X, Y \in \mathcal{C}$, the following chain stabilizes:

$$\mathfrak{I}_0(X, Y) \supseteq \mathfrak{I}_1(X, Y) \supseteq \mathfrak{I}_2(X, Y) \supseteq \dots$$

- The transfinite radical of $\mathrm{mod}\,\Lambda$ is often non-zero, for example for:
  - "Gelfand-Ponomarev" algebra $k[x,y]/(xy = yx = 0, x^2 = y^3 = 0)$ [Prest].

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There is a direct translation between the two settings for self-injective artin algebras [Krause-Solberg] and [Angeleri-Šaroch-Trlifaj].

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Telescope conjecture

Given a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B}$ is closed under direct limits, $(\mathcal{A}, \mathcal{B})$ is of finite type.

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Theorem (Šaroch-Š.)

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Let Λ be an artin algebra and $(\mathcal{A}, \mathcal{B})$ a hereditary cotorsion pair such that $\mathcal{B}$ is closed under direct limits.
Let $\mathfrak{I}$ be the ideal of $\operatorname{mod} \Lambda$ defined as:
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- Statement 2. is a consequence of approximation theory and model theory of modules.

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Let Λ be an artin algebra such that the transfinite radical of $\operatorname{mod}\Lambda$ vanishes. Let $(\mathcal{A}, \mathcal{B})$ a hereditary cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Then $(\mathcal{A}, \mathcal{B})$ is of finite type.

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- Hence, $\mathcal{B}$ is of finite type.

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