

*Idempotent ideals in a module category and
the Telescope conjecture*

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Notation

For an ideal \mathfrak{I} , let $\mathfrak{I}(X, Y) = \{f : X \rightarrow Y \mid f \in \mathfrak{I}\}$.

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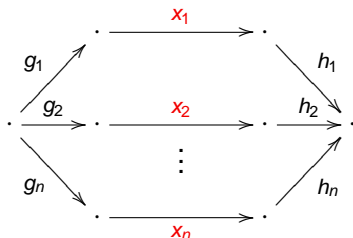
An ideal \mathfrak{I} is **idempotent** if $\mathfrak{I}^2 = \mathfrak{I}$. Equivalently:

$$(\forall f \in \mathfrak{I})(\exists g, h \in \mathfrak{I}) \quad f = gh$$

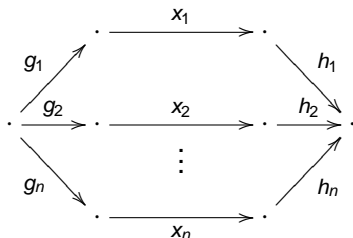
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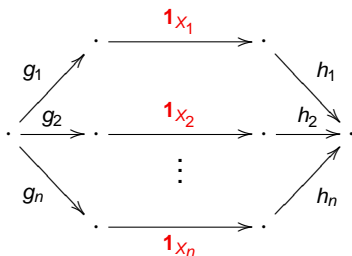


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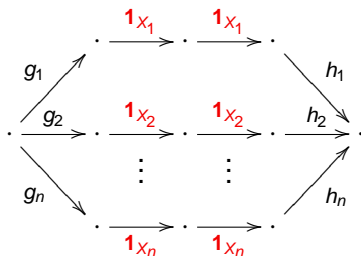
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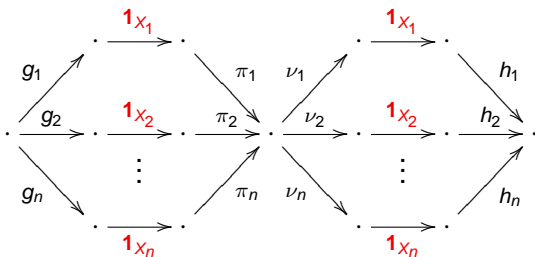
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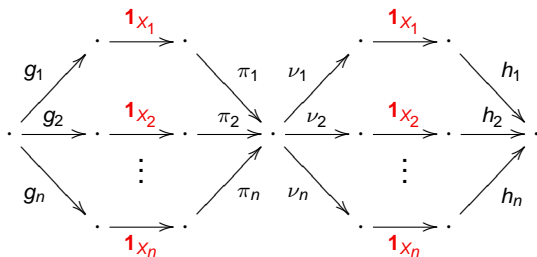
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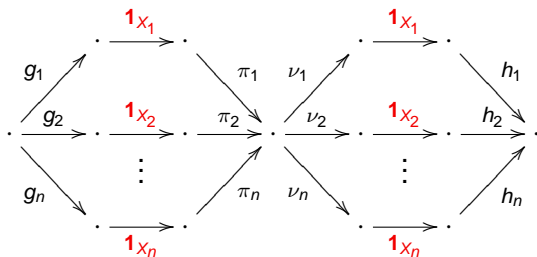
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- ▶ Often **not**, but it is true for some non-trivial cases.

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Definition “from inside”: The ideal of all morphisms f such that there exist an inverse system

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 - ▶ By assumption: $h_\alpha = 0$ for $\alpha \gg 0$.

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$$\mathfrak{I}_0(X, Y) \supseteq \mathfrak{I}_1(X, Y) \supseteq \mathfrak{I}_2(X, Y) \supseteq \dots$$

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There is a direct translation between the two settings for self-injective artin algebras [Krause-Solberg] and [Angeleri-Šaroch-Trlifaj].

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Telescope conjecture

Given a hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ such that \mathcal{B} is closed under direct limits, $(\mathcal{A}, \mathcal{B})$ is of finite type.

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- ▶ First show that \mathcal{B} is of **countable type** (set-theoretic methods, properties of cotorsion pairs and inverse limits).

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- ▶ First show that \mathcal{B} is of countable type (set-theoretic methods, properties of cotorsion pairs and inverse limits).
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Theorem (Šaroch-Š.)

Let Λ be an artin algebra and $(\mathcal{A}, \mathcal{B})$ a hereditary cotorsion pair such that \mathcal{B} is closed under direct limits.

Let \mathfrak{J} be the ideal of $\text{mod } \Lambda$ defined as:

$\mathfrak{J} = \{f : X \rightarrow Y \mid f \text{ factors through some (inf. gen.) module from } \mathcal{A}\}$

Then:

1. \mathfrak{J} is an idempotent ideal.
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- ▶ Statement 2. is a consequence of approximation theory and model theory of modules.

Theorem

Let Λ be an artin algebra such that the transfinite radical of $\text{mod } \Lambda$ vanishes. Let $(\mathcal{A}, \mathcal{B})$ a hereditary cotorsion pair such that \mathcal{B} is closed under direct limits. Then $(\mathcal{A}, \mathcal{B})$ is of finite type.

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- ▶ Hence, \mathcal{B} is of finite type.

