Universal Algebra 1 - Homework 4

Deadline 20.12.2018, 10:40

1. (6 points) Let $\mathcal{V}$ be the variety of algebras $(A, \cdot, l, r)$ of type $(2, 1, 1)$ that satisfy the identities
$$l(x \cdot y) \approx x, \quad r(x \cdot y) \approx y, \quad l(x) \cdot r(x) \approx x.$$
   (a) Show that every non-trivial member of $\mathcal{V}$ is infinite.
   (b) Prove that, if $\mathbf{A} \in \mathcal{V}$ is generated by $\{a_1, a_2, \ldots, a_n\}$, then it is already generated by $\{(a_1 \cdot a_2), a_3, \ldots a_n\}$
   (c) Prove that $\mathbf{F}_{\mathcal{V}}(n) = \mathbf{F}_{\mathcal{V}}(m)$ for all positive integers n, m.

2. (6 points) Let $\mathbf{A}$ be the algebra given by the following multiplication table:

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

   Prove that the variety generated by $\mathbf{A}$ is exactly the variety of commutative semigroups satisfying $x^2 \approx x^3$.

3. (8 points) Let $\mathcal{V}$ the variety of algebras $(A, \cdot)$ satisfying the identities
$$x \cdot x \approx x \text{ and } (x \cdot y) \cdot z \approx (z \cdot y) \cdot x.$$
   (a) Show that every member of $\mathcal{V}$ also satisfies the identities
$$\begin{aligned}(x \cdot y) \cdot (z \cdot w) &\approx (x \cdot z) \cdot (y \cdot w)\\ x \cdot (y \cdot z) &\approx (x \cdot y) \cdot (x \cdot z)\\ (y \cdot z) \cdot x &\approx (y \cdot x) \cdot (z \cdot x)\\ y \cdot (x \cdot y) &\approx (y \cdot x) \cdot y\\ (y \cdot x) \cdot x &\approx x \cdot y\end{aligned}$$
   (b) Let $\mathcal{W}$ be the subvariety of $\mathcal{W}$ defined by the additional identity $y \cdot (x \cdot y) \approx x$. Determine $\mathbf{F}_{\mathcal{W}}(x, y)$ (multiplication table).