# Universal Algebra 1 - Homework 3

Deadline 6.12.2018, 10:40

1. (6 points) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups. Define $\mathcal{V} \cdot \mathcal{W}$ to be the class of all groups $\mathbf{A}$ containing a normal subgroup $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{V}$ and $\mathbf{A}/\mathbf{B} \in \mathcal{W}$. Prove that $\mathcal{V} \cdot \mathcal{W}$ is a variety.

2. (6 points) Let $\mathbf{L}, \mathbf{M}$ be two non-trivial lattices. By $\mathbf{L} \oplus \mathbf{M}$ we define the lattice with universe $L \dot{\cup} M$ such that every element of $L$ lies below every element on $M$. Prove that $\mathsf{HSP}(\{\mathbf{L}, \mathbf{M}\}) = \mathsf{HSP}(\{\mathbf{L} \oplus \mathbf{M}\})$ (hint: subdirect representation)

3. (8 points) Let $\mathbf{R}$ be a commutative ring with 1. Further assume that $x \neq 0 \leftrightarrow \forall n : x^n \neq 0$ holds in $\mathbf{R}$ (such a ring is also called *reduced*).

   - Show that for every $a \in R \setminus \{0\}$ there is a prime ideal $P_a$ with $a \notin P_a$ (Hint: Show that there is a maximal ideal that excludes $a, a^2, a^3, \ldots$)
   - Prove that $\mathbf{R}$ is the subdirect product of integral domains.