$(2) \Rightarrow (3)$ $\quad a \,\mathcal{L}\, a'a = aa' \,\mathcal{R}\, a$ $\rightsquigarrow$ idempotent in H_a

$(3) \Rightarrow (4)$ Green's theorem

$(4) \Rightarrow (1)$ $b := a^{-1}$ in the group H_a

__Coro__: S __completely regular__ $\Leftrightarrow$ $\exists$ unary op $'$ s.t. $\boxed{\begin{matrix} xx'x = x \\ x'' = x \\ xx' = x'x \end{matrix}}$

completely regular $\Leftrightarrow$ $\forall a \in S$ is comp. reg.

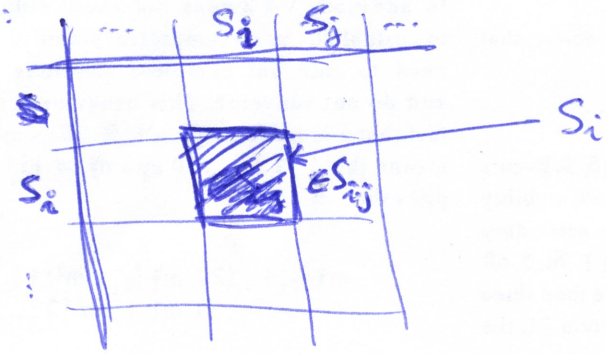
__Pf__: ($\Rightarrow$) $x' :=$ the unique elt. from (2)

($\Leftarrow$) x' satisfies (1)

__Construction__ (Lallement): __band of semigroups__: I idemp. semigroup, $S_i : i \in I$ semigroups

$\rightsquigarrow$ __b. of s.__ is $S = \dot{\bigcup}_{i \in I} S_i$ s.t. $S_i \cdot S_j \subseteq S_{i \cdot j}$

... __Cayley table__:



... __decomposing congruence__: $\sigma := \{(a,b) : \exists i \; a,b \in S_i\}$

ⓞ $\sigma \in \mathrm{Con}\, S$ ... $a \sigma b \Rightarrow \exists i \; a,b \in S_i$

$\Rightarrow \forall c \in S_j \quad ac, bc \in S_{ij}$
$\qquad\qquad\quad ca, cb \in S_{ji}$

$\Rightarrow ac \,\sigma\, bc, \; ca \,\sigma\, cb$

ⓞ $S/\sigma \simeq I$

... $\alpha \in \mathrm{Con}\, S$ s.t. S/α idempotent $\Rightarrow$ $S \simeq$ band of smg. s.t. $\left[\begin{matrix} I = S/\alpha \\ S_i \text{ are blocks} \end{matrix}\right.$