Using Lallement's lemma :

Ex.: $\left.\begin{array}{l} S \text{ inverse} \\ \alpha \in \operatorname{Con} S \end{array}\right\} \Rightarrow \; S/\alpha \text{ is a group} \iff \operatorname{tr} \alpha = E_S^2$

Pf: Note: T inverse $\Rightarrow \left[T \text{ is a group} \iff |E_T| = 1 \right]$
(look at eggbox)

... ~~Lallement~~ Lallement: S/α is inverse

$$\Rightarrow \left[S/\alpha \text{ group} \iff |E_{S/\alpha}| = 1 \iff \operatorname{tr} \alpha = E_S^2 \right]$$

($\Rightarrow$) obvious

($\Leftarrow$) let $[a] = E_{S/\alpha}$
$\quad \| \text{Lallement}$
$\quad [e]$ for some $e \in E_S$
all congruent

$\square$

Another formulation of Lallement's lemma :

L.ℓ.^{H}: S regular, $\varphi : S \twoheadrightarrow T$ hom., $f \in E_T \Rightarrow \exists e \in E_S$ s.t. $\varphi(e) = f$

Coro.: S inverse, $\varphi : S \twoheadrightarrow T$ hom. (w.r.t. $\circ$) $\Rightarrow T$ inverse & $\varphi(x') = \varphi(x)'$ $\forall x \in S$

Proof: 1^{st} iso. thm: $T \cong S/\ker \varphi$
$\varphi(x) \leftrightarrow [x]$

... L.ℓ.^{H}: $\exists a \in S$ s.t. $\varphi(a) = f$
by L.ℓ.^{C}: $\exists e \in E_S$ $\underbrace{(a,e) \in \ker \varphi}_{f = \varphi(a) = \varphi(e)}$

... Coro: $\ker \varphi \in \operatorname{Con}(S)$, hence $T \cong S/\ker \varphi$ is inverse

notice that $\left.\begin{array}{l} \varphi(x) = \varphi(xx'x) = \varphi(x)\varphi(x')\varphi(x) \\ \varphi(x') = \varphi(x'xx') = \varphi(x')\varphi(x)\varphi(x') \end{array}\right\}$ unique inverses $\Rightarrow \varphi(x') = \varphi(x)'$

$\square$