Binary systems - homeworks

More interesting exercises are marked with [!].
To pass the course, solve at least half of the exercises marked [!] (i.e., nine).

1. Introduction

1. [!] a) Find a binary algebra $(A, \cdot)$ that is latin, but has a non-latin subalgebra and a non-latin factoralgebra. b) Show that there is no finite example for a). c) Consider a latin binary algebra with the division operations, i.e., $(A, \cdot, /, \backslash)$. Show that all subalgebras and factoralgebras are latin too.

2. [!] Let A, B be two latin binary algebras and $\varphi : A \to B$ a homomorphism (with respect to $\cdot$). Show that φ preserves the division operations, too.

3. [!] Prove that there is 1-1 correspondence between Steiner triple systems, and Steiner quasigroups, i.e., binary systems $(A, \cdot)$ satisfying $xx = x$, $x(xy) = y$, $xy = yx$.

4. Consider the 16-element loop of octonion units (i.e., the elements are ± 1, $\pm e_1$, ..., $\pm e_7$). Show that it is not associative, but it has the property that every subloop generated by two elements is associative. Try to prove that this loop satisfies one of the Moufang identities (disclaimer: I don't know if this is too difficult or if there is a simple argument - think about it yourself! :-))

5. Show that left distributive left quasigroups satisfy $x\backslash(yz) = (x\backslash y)(x\backslash z)$ and $x(y\backslash z) = (xy)\backslash(xz)$.

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2. Semigroups

6. Show that the semigroup of words is free. (The alphabet is the free base.)

7. Show that semilattices are really associative.

8. [!] Consider a monogenerated subsemigroup $\langle a \rangle$ with preperiod n and period p. Show that $\{a^n, a^{n+1}, \dots, a^{n+p-1}\}$ is a group.

9. [!] Show that, in T_X,

a) $f \leq_{\mathcal{R}} g$ iff $\mathrm{rng}(f) \subseteq \mathrm{rng}(g)$,

b) $f \,\mathcal{D}\, g$ iff $f \,\mathcal{J}\, g$ iff $|\mathrm{rng}(f)| = |\mathrm{rng}(g)|$.

Is $\mathcal{D} = \mathcal{J}$?

10. Determine the $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ relations in the matrix semigroup $M_n(F)$. Is $\mathcal{D} = \mathcal{J}$?

11. Determine the $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ relations in the symmetric inverse semigroup I_X. Is $\mathcal{D} = \mathcal{J}$?

12. Show that, in the matrix semigroup

$$\left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a, b > 0 \right\},$$

we have $id = \mathcal{D} \neq \mathcal{J}$.

13. [!] Show that a semigroup is a group if and only if it contains no proper left ideals and no proper right ideals (*proper* means strictly smaller than S). Hint: use Green's theorem.

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14. [!] Show that the following are equivalent for a regular semigroup:

(1) it has exactly one idempotent;
(2) it is cancellative (i.e., $xu = xv$ implies $u = v$, and $ux = vx$ implies $u = v$);
(3) it is a group.

15. [!] Determine whether a) PT_X (partial transformations), b) B_X (binary relations), c) $M_n(F)$ ($n \times n$ matrices over F) are regular semigroups.

16. [!] Show that $L_a \leftrightarrow R_{a'}$ is a bijection between $\mathcal{L}$-blocks and $\mathcal{R}$-blocks in a given $\mathcal{D}$-block in every inverse semigroup. (Hence the number of $\mathcal{L}$-blocks equals the number of $\mathcal{R}$-blocks.)

17. Show that $(S, \cdot, ')$ is an inverse semigroup iff for every $x, y, z \in S$

$$x(yz) = (xy)z,\ xx'x = x,\ x'xx' = x',\ xx'yy' = yy'xx'.$$

We did ($\Rightarrow$) at the lecture. For ($\Leftarrow$), the hint is to prove that $e = ee'$ for every e idempotent.

18. Show carefully that I_X is an inverse semigroup. (I am not sure I did it carefully at the lecture.)

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19. [!] Finish the proof of the theorem relating congruences and congruence pairs.

20. Show that the intersection of congruences corresponds to the intersection of congruence pairs (you intersect both N's and τ's).

21. Think about the smallest and the largest congruence with a given a) trace, b) kernel. This is not an easy exercise, but the result is useful. (See Howie for hints or solution.)

22. Let S be an inverse semigroup. Show that there is the smallest congruence σ such that S/σ is a group. Calculate its kernel and trace. Show that $\alpha \vee \sigma = \sigma \circ \alpha \circ \sigma$ for every congruence α of S.

Open problem. Describe the center congruence in inverse semigroups. Characterize abelian congruences in inverse semigroups. (See universal algebra II for definitions.)

23. Show carefully that $t(x, y, z, u, v) = xy'zu'v$ is a Taylor term for the variety of inverse semigroups. Show that semigroups are not congruence modular (hint: find a semilattice violating the property) and not meet-semidistributive (hint: find a group violating the property). Advanced exercise: show that inverse semigroups have a weak difference term (if you know what it is).

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24. Show that $\alpha \in PT_X$ is a completely regular element iff $\alpha|_{\text{rng}\alpha}$ is a permutation.

25. In completely regular semigroups, for every element a, there is a unique inverse a' to a with the property that $aa' = a'a$. We can think about $'$ as a basic operation. Show that $(S, \cdot, ')$ is a completely regular semigroup iff for every $x, y, z \in S$

$$x(yz) = (xy)z, \ xx'x = x, \ x'xx' = x', \ xx' = x'x.$$

26. Show that, in a completely regular semigroup S, $\mathcal{J}$ is a congruence, and the factor $S/\mathcal{J}$ is a semilattice. Hint: start with the proof that $a\mathcal{J}a^2$ and $ab\mathcal{J}ba$ for every $a, b \in S$.

27. Show that a semigroup is a band of groups if and only if it is completely regular and $\mathcal{H}$ is a congruence. (A band is an idempotent semigroup.)

28. [**!**] Show that Rees matrix semigroups are completely simple.

29. [**!**] Let F be a field and let

$$S = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} : A \text{ is a regular } n \times n \text{ matrix over } F, \ b \in F^n \right\}$$

be a subsemigroup of the matrix semigroup $M_{n+1}(F)$. Show that S is completely simple and find its Rees matrix representation.

30. [**!**] Show that the following conditions are equivalent for any semigroup S.

(1) S satisfies $xyx = x$ for every $x, y \in S$.
(2) S is idempotent and satisfies $xyz = z$ for every $x, y, z \in S$.
(3) S is the direct product of a left projection semigroup and a right projection semigroup.

(Such semigroups are called rectangular bands.) Hint for (2) $\Rightarrow$ (3): let $L = Sa$ and $R = aS$ for a fixed a.

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3. Mediality and quandles

31. [!] Find a quasigroup that is isotopic to two non-isomorphic loops.

32. I failed to finish the proof of Toyoda's theorem. Read the rest of it in my notes (pages 5,6).

33. I also put in my notes the idea how to prove Toyoda's theorem using the Gumm-Smith theorem on abelian algebras (from universal algebra). If you attend the Universal Algebra II course, read it and try to work the details (pages 6,7).

34. [!] For an abelian group $(A,+)$ and an automorphism φ, denote $Q(A,\varphi)$ the medial quasigroup $(A,*)$ with $x*y=(1-\varphi)(x)+\varphi(y)$ (here 1 denotes the identity mapping). Show that two medial quasigroups $Q(A,\varphi)$ and $Q(B,\psi)$ are isomorphic if and only if there is a group isomorphism $\rho:(A,+)\to(B,+)$ such that $\psi=\rho\varphi\rho^{-1}$.

35. Using the previous exercise, calculate the number of idempotent medial quasigroups of size a) p, b) p^2, up to isomorphism (for any prime p).

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36. [!] Prove carefully that 3-coloring is an invariant with respect to the Reidemester move III. (I did moves I and II at the lecture.)

37. [!] Show that there is no non-trivial 3-coloring of the figure-8 knot. Find a quandle such that there is a non-trivial coloring of this knot. (Coloring is trivial, if it uses only one color.) Hint: a connected affine quandle, i.e., an idempotent medial quasigroup, will work. 3-coloring uses $Q(\mathbb{Z}_3,2)$. Try a slightly bigger group.

38. [!] Show that Galkin quandles are really quandles.

39. Prove the Galkin's representation theorem for connected quandles. (Hints were at the lecture.)