A SURVEY OF LEFT SYMMETRIC LEFT DISTRIBUTIVE GROUPOIDS

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Note. *The paper is in progress (probably forever) and may contain inaccurate citations, so I strongly recommend to check original papers, whenever a reader becomes interested in a particular result. Also, I believe I found almost all English-written references dealing with LSLDI groupoids, however, the survey is far from making a big picture about them, it is rather just overview of papers related to the topic. The stress on particular results depends on my mood in time I held the respective paper and it is far from being balanced (and I doubt the topic is worth of the effort to make the survey nice and complete).*

We present a survey of results about groupoids, i.e. binary algebras, satisfying the following identities:

(left symmetry) $x(xy) \approx y,$

(left distributivity) $x(yz) \approx (xy)(xz),$

(idempotency) $xx \approx x.$

We call such groupoids *left symmetric left distributive idempotent*, shortly *LSLDI*, while various other names were used by former authors: *kei* by M. Takasaki, *symmetric sets* by N. Nobusawa, *symmetric groupoids* by R. S. Pierce, *involutory quandles* by D. Joyce. Note that a groupoid G is left symmetric, iff every mapping $L_a : G \to G$, $x \mapsto ax$, called the *left translation by* a, is either the identity, or an involution (in other words, permutation of order 2). It is left distributive, iff every left translation is an endomorphism of G.

At the end, we mention briefly two more general classes of LD groupoids: *LDI left quasigroups* (called by geometers also racks, wracks, quandles, crystals, automorphic sets, etc.) and *LSLD (generally non-idempotent) groupoids.*

The first explicit allusion to selfdistributivity is, perhaps, in the work of C. S. Peirce [Pe] from 1880. He discusses various forms of a distributive law and he pleads also that selfdistributivity, *"which has hitherto escaped notice, is not without interest."* Another early work, which mentions an example of a non-associative distributive structure, is [Sc] of E. Schröder (1887). The first known article fully pursued to selfdistributive structures is [BM] of C. Burstin and W. Mayer from 1929. They investigated two-sided distributive idempotent quasigroups. The first articles dealing with non-idempotent distributive groupoids were [Ru] of J. Ruedin

(1966) in the two-sided case and [Ke1] of T. Kepka (1981) in the one-sided case. A comprehensive survey for study of two-sided selfdistributive systems is [JKN], regarding one-sided distributivity we recommend a recent book [De] of P. Dehornoy.

The first paper on left symmetric one-sided distributive groupoids is probably [Ta] of M. Takasaki (1943), however most results come from 70's and 80's. They appear in a bunch of short notes and they are mostly fragmentary. Non-idempotent LSLD groupoids were started in [Ke2] by T. Kepka (1994).

We use terminology and notation usual in universal algebra. A standard reference is a textbook [BS]. For groupoid terminology see also [KN]. We often use contemporary terminology instead of the original one.

1. Geometrical motivation

The origin of the theory of LSLDI groupoids lies in the book [Lo] of Ottmar Loos (1969). He studied so called *symmetric spaces* and he found that they can be described equivalently as differentiable manifolds equipped with a smooth LSLDI operation satisfying the following condition: for any point a, there is a neighborhood U of a such that a is the only fixed point of the left translation L_a on U.

A basic example of a symmetric space comes as follows: on any manifold with metric, one can define a product $x \circ y$ to be the image of y by the symmetry through x. For instance, on the real line $x \circ y = 2x - y$, on the n-sphere $x \circ y = 2\langle x, y\rangle x - y$. Actually, the 3-sphere gives a reason why the fixed point condition is defined locally — the point symmetric to the south pole through the north pole is the south pole. A different example is the set of all $n \times n$ positive definite symmetric matrices over the reals with the operation $A \circ B = AB^{-1}A$. Grassman manifolds and Jordan algebras are also symmetric spaces. And, of course, any LSLDI groupoid can be considered as a discrete symmetric space.

Motivated by the above natural example, Nobuo Nobusawa [No1] and others started to investigate LSLDI groupoids from a purely algebraic point of view in 70's. Moreover, in 80's, D. Joyce found an important application of LSLDI groupoids in the knot theory (see a later section).

2. Algebraic examples

Small examples. The following are all (up to an isomorphism) two-element LSLD groupoids (one idempotent, the other not).

	a	b
a	a	b
b	a	b

	a	b
a	b	a
b	b	a

The following are all (up to an isomorphism) three-element LSLDI groupoids. The third one is a commutative distributive quasigroup and it forms the smallest Steiner triple system. It appears as an example of a non-associative distributive structure already in the paper [Sc] (1887).

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

	a	b	c
a	a	c	b
b	a	b	c
c	a	b	c

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

RZB. We can define an LSLDI groupoid on every set G: put $a \circ b = b$ for all $a, b \in G$. The groupoid $G(\circ)$ is called *right zero band*. All $G(\circ)$, G a set, form a minimal subvariety of LSLDI. Right zero bands are the only associative LSLDI groupoids.

Cores of groups. Let G be a group. We define a new operation $*$ on G by

$$a * b = ab^{-1}a.$$

It is easy to check that the resulting groupoid $G(*)$, called the *core* of G, is LSLDI. Moreover, the variety of LSLDI groupoids is generated by all $G(*)$, G a group (see [Pi1]). The core operation was introduced in a book [Br] of R.H. Bruck more generally for loops, by

$$a * b = a(b \backslash a).$$

He used them as invariants of isotopy. Later, it was observed that cores of (left) Bol loops are LSLDI. (Left Bol loop is a loop, i.e. a "non-associative group", satisfying $x(y(xz)) \approx (x(yx))z$.) Note that cores of Lie groups (or generally smooth Bol loops) are symmetric spaces.

For more constructions of LSLD groupoids from groups see [St1].

Medial groupoids. Cores of abelian groups satisfy an identity

(mediality) $$(xy)(uv) \approx (xu)(yv),$$

in some papers called *entropy*. It is easy to see that mediality and idempotency imply both left and right distributivity. The variety of LSMI groupoids was extensively studied in 80's and 90's by B. Roszkowska, who called them *SIE groupoids*.

3. Structural results

The contents of the first (known to us) paper [Ta] of M. Takasaki on LSLDI groupoids, is unfortunately a big puzzle — it is in Japanese. The paper is rather long and it seems to contain some basic properties and many examples.

Some of the results mentioned in this section are reproven in the author's PhD Thesis [St2].

First steps. Nobuo Nobusawa, involved by O. Loos, began to study LSLDI groupoids from a purely algebraical point of view. In his first paper [No1] (1974), he defined a *symmetric set* to be a mapping S from a finite set A to the symmetric group over A, sending an element a onto an involution S_a such that $S_a(a) = a$ and $S_{S_a(b)} = S_a S_b S_a$ for all $a, b \in A$. Clearly, setting $a \circ b = S_a(b)$, we obtain an LSLDI groupoid $A(\circ)$, in which S_a is the left translation of a. (Actually, all japanese papers use the dual, i.e. right symmetric right distributive, notation.) Cores of groups are shown as an example. Nobusawa defines a *group of displacements* $D(A)$ of a symmetric set A, to be the subgroup of the symmetric group over A generated by all $S_a S_b$, $a, b \in A$. This group plays an important role in investigation of the structure of LSLDI groupoids. For example, it is easy to see that an LSLDI groupoid is medial (called *abelian* by Nobusawa), iff its group of displacements is abelian. Also, an *effective* (i.e. such that $S_a \neq S_b$ whenever $a \neq b$) LSLDI groupoid A is medial, iff its group of displacements consists of all $S_e S_a$, $a \in A$, where $e \in A$ is fixed. In such a case, A is a quasigroup (originally *homogeneous*). Note that $a \mapsto S_e S_a$ is a homomorphism of A into the core of $D(A)$, for any $e \in A$. A power of an element a is defined to be $a^k = \underbrace{e(a(e(a\dots)))}_{k+1}$, thus *cycles* are defined. (Note that a cycles are homomorphic images of the core of the group of integers.) A finite LSLDI groupoid is a quasigroup, iff all cycles are of odd length. All LSLDI quasigroups on p^2 elements ($p \geq 3$ prime) are cores of abelian groups, however, an example of a 27-element LSLDI quasigroup which is not core of a group is given. At the end, there is a list of all at most 5-element LSLDI groupoids (recently, A. Vanžurová pointed out that certain two of them are isomorphic).

Another early article [KNN] is devoted to homogeneous symmetric sets, i.e. finite LSLDI quasigroups. Among others, he shows that an effective LSLDI groupoid is medial, iff it is a core of an abelian group, and the core of a finite group is a quasigroup, iff it is of odd order. Further, an analogy of Lagrange's theorem is proven: if B is a subgroupoid of a finite LSLDI quasigroup A, then $|A|$ is divisible by $|B|$.

N. Umaya in [Um] investigates cores of groups. The core of a group G is effective, iff the center of G does not contain an involution. The group of displacements $D(G)$ of the core of a group G is generated by all $L_g R_g$, $g \in G$, where L_g, R_g are left/right translations by g in the group G. Suppose for the rest of the paragraph that the center of G is trivial. An immediate consequence of the previous statement is that $D(G)$ can be embedded into the product of $L(G) \times R(G)$, where $L(G), R(G)$ are groups generated by left/right translations of G. Moreover, $D(G)$ consists of all $L_b L_a R_{b'} R_a$, $a \in G$, $b, b' \in G'$, where G' denotes the commutator subgroup. If N is a $D(G)$-orbit in G containing the unit, then N is a normal subgroup of G and G/N is an elementary abelian subgroup of exponent 2. Consequently, if $G = G'$, then $D(G) \simeq L(G) \times R(G)$.

Simple LSLDI groupoids. The short note [No2] of Nobusawa shows that the group of displacements of a simple LSLDI groupoid A (not necessarily finite) is a minimal normal subgroup of the symmetric group over A. Hence, it is either simple or a direct product of two simple groups, which are conjugated. Moreover, under the second condition, the group of displacements has precisely $|A|^2$ elements (and A is primitive, as shown in [Na]). The article continues with an application of the result on the structure of symmetric groups.

Nobusawa's second paper [IN] concerning simplicity (with Y. Ikeda) discusses subgroupoids of the core of $SL_n(F)$, F a field. E.g., they state that the group of displacements of the core of $SL_n(F)$ is isomorphic to $SL_n(F)/\{\pm 1\}$ for $n \geq 3$ or for $n = 2$ and $F \neq \mathbb{Z}_3$. The same holds for $PSL_n(F)$ too. Further, the core of $SL_n(\mathbb{Z}_q)$, q being a power of a prime p, is left ideal-free (originally *transitive*), iff $p \neq 2$ or n is odd; otherwise it consists of precisely two disjoint left ideals. Many examples are shown.

Next contribution is [Na] by H. Nagao. He discussed transitive and primitive symmetric sets (in a sense of the action of the group generated by left translations; hence, transitive means left ideal-free and primitive means transitive and there is no subset $I \subset A$ such that $AI \cap I = \emptyset$). Every primitive LSLDI groupoid is simple and every simple LSLDI groupoid is transitive. If G is a group generated by a set A of involutions in G such that A is a conjugation class of G and the subgroup generated by AA is a minimal normal subgroup of G, then the set A with the core operation is a simple LSLDI groupoid.

In [No5] Nobusawa applies the developed theory to find a new proof (via symmetric sets) that orthogonal groups are simple.

A characterization of finite simple LSLDI groupoids was found by D. Joyce in [Jo2]; in fact, he characterized all simple LDI left quasigroups. A finite LSLDI groupoid is simple, iff it is isomorphic either to the core of a simple group, or to a conjugation class of involutions in a simple non-abelian group (with the core operation).

R. S. Pierce. A big portion of structure results was developed by R. S. Pierce in [Pi1] and [Pi2]. Both papers are very comprehensive and we refer a reader mainly to the original text. The fundamental idea is a correspondence between LSLDI groupoids and groups generated by involutions. The fundamental notion is the subgroupoid $I(G) = \{x \in G : x^2 = 1\}$ of the core of a group G. Among others, Pierce describes free LSLDI groupoids, he investigates properties of $I(G)$, cycles (cf. the first Nobusawa's paper), some theorems on decomposition to disjoint left ideals are proven. *Balanced* LSLDI groupoids (satisfying $xy = y$ iff $yx = x$; note that cores are balanced) are studied using a graph theoretical method.

More geometrical examples. The paper [No3] of Nobusawa is devoted to the following LSLDI groupoid. Let V be a vector space over a field F, $\mathrm{char}(F) \neq 2$, and let g be a non-degenerate symmetric bilinear form on V. Let $A = \{a \in V : g(a, a) = 1\}$. For $a \in A$ we put L_a to be the reflection in the hyperplane orthogonal to a. Now, let B be the set of all 1-dimensional subspaces of V generated by an element of A. We equip the set B with an operation $\langle a\rangle \circ \langle b\rangle = L_a(\langle b\rangle)$. Then $B(\circ)$ is an LSLDI groupoid and Nobusawa describes its properties. E.g. it is left ideal-free, if there are a, b linearly independent such that $g(a, a) = g(b, b) \neq 0$. It is primitive, if $\dim V > 4$ or if $\dim V = 4$ and $F \neq \mathbb{Z}_3$.

The paper [En2] of N. Endres from 90's contains a bunch of other examples, mainly geometrical. Endres investigated in his earlier papers certain operations on homotopy classes on spheres *(???)*, which turn to be LSLDI. *To be added.*

Further results. The paper [Ki1] of M. Kikkawa establishes a connection between LSLDI quasigroups and a certain class of loops, called *symmetric loops*. He proves that the operation $a \circ b = R_e^{-1}(a)L_e(b)$ defined on an LSLDI quasigroup (e is a fixed element) yields a symmetric loop and the operation $a * b = a^2b^{-1}$ yields for every symmetric loop an LSLDI quasigroup. Moreover, the two functors are mutually inverse. It can be shown that symmetric loops are exactly what is now known as Bruck loops of odd order (see [St2]) or B-loops by Glauberman (see [Gl]). The Kikkawa's connection is more or less the same one developed independently by the author in [St2]. Some of the Kikkawa's results were also found independently by Nagy and Strambach in [NS] (they use LSLDI groupoids when investigating smooth Bol loops). The consecutive papers [Ki2] and [Ki3] of M. Kikkawa develop some further properties of symmetric loops and other similar classes of loops.

The paper [Jo1] of D. Joyce about quandles and their application in knot theory contains also a couple of results on LSLDI groupoids (called involutory quandles there). See a later section.

A. Vanžurová investigates in [Va] hypersubstitutions in the variety of LSLDI groupoids.

J. Ježek and T. Kepka prove the following two general results with a non-trivial application in the left symmetric case. A distributive idempotent groupoid A is symmetric-by-medial, which means there exists a congruence $\sim$ of A such that the factor $A/\sim$ is medial and every block of $\sim$ is symmetric and commutative (see [JK1]). A non-medial (left and right) distributive groupoid with all proper subgroupoids medial is a quasigroup; consequently, a non-medial distributive groupoid has at least 81 elements (see [JK2]).

S. Doro uses LSLDI groupoids in [Do] when investigating simple Moufang loops.

4. Russian school

There are probably papers in Russian refering to LSLDI groupoids, but I have no idea how to find them. The search is to be continued. First achievements see below.

It seems to me that most of the results are closely related to quasigroups. It means, either cores of loops, or LDI quasigroups are considered.

The definition of the core of a Moufang loop appears in the book [Br] of R. S. Bruck. However, V. D. Belousov was probably the first who studied cores of groups and loops as examples of LSLDI groupoids. The whole chapter is devoted to cores of loops in his book [Be2]. The paper [Be1] of Belousov contains the following result: the core of a group G is right distributive, iff $xy^2x \approx yx^2y$ holds in G. A classical result of Belousov is characterization of commutative LSLDI groupoids (in other words, distributive symmetric quasigroups): they are precisely cores of commutative Moufang loops satisfying $3x \approx 0$.

Galkin in [Ga1] shows that finite LSLDI quasigroups are *solvable*, that means they can be constructed by a chain of extensions by medial quasigroups. Moreover, an LSLDI quasigroup G (not necessarily finite) is solvable, iff for every φ in the commutant of the left multiplication group of G and for every $a \in G$ the mapping $L_a\varphi$ has a unique fixpoint. An example of a non-solvable (infinite) LSLDI quasigroup is provided: on the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1, x_3 > 0\}$ take the operation $x \circ y = 2\langle x, y\rangle x - y$, where $\langle x, y\rangle = -x_1y_1 - x_2y_2 + x_3y_3$.

In his another paper [Ga2], Galkin discusses the question, whether finite left distributive quasigroups possess Sylow property. He proves that an LDQ G has Hall subquasigroups, if G is solvable and there is a number m relatively prime to $|G|$ such that $L_a^m = id$ for every $a \in G$. It is easy to see that the conditions are satisfied in any finite LSLDI quasigroup (which have always odd size).

5. Left symmetric medial idempotent groupoids

Finite LSMI groupoids were described by B. Roszkowska in a series of papers [Ro1], [Ro2], [Ro3], [Ro4]. They can be constructed from cores of abelian groups by certain construction. She described also all subvarieties of LSMI groupoids. They form a lattice isomorphic to the lattice of integers under division with an added top element.

A special class of LSMI groupoids, so called group related symmetric groupoids, were studied by N. Endres in [En1] and their (non-symmetric and eventually also non-distributive) generalization in [En3]. Hyperidentities in LSMI groupoids were studied by Arworn and Denecke in [AD]. An application of LSMI groupids to graph theory can be found in [Ro2].

Certain subclass of LSMI groupoids was studied in [RR1]: they described LSI groupoids satisfying $x \cdot yz \approx y \cdot xz$ and $xy \approx zx \cdot y$. A non-symmetric generalization was studied in [RR2] and [Pl]: instead of left symmetry an equation $x(x(\dots(xy))) \approx y$ is considered (such groupoids were called n-cyclic). E.g., if $G(+)$ is an abelian group with $mn^2x \approx 0$, then $x \circ y = (1-mn)x + mny$ is a n-cyclic groupoid. All n-cyclic groupoids can be constructed from abelian groups by certain construction. The only subvarieties of n-cyclic groupoids are m-cyclic groupoids with $m|n$.

6. LDI left quasigroups

A groupoid is called a *left quasigroup* (shortly LQ), if all left translations are permutations. Clearly, left symmetric groupoids are left quasigroups.

LD(I) left quasigroups are known under a bunch of names, such as racks, wracks, quandles, automorphic sets, pseudo-symmetric sets, crystals, etc. (some of the names refer to the idepotent case, some don't). They are rather widely studied, mainly because of their geometrical applications. We list some basic references.

Of course, I must recommend my PhD Thesis [St2] as the best (or at least the most recent) source of information :–)

The paper [FR] of R. Fenn and C. Rourke (1992) summarizes the use of LDLQ's in knot theory. It contains both an algebraic theory (more or less a survey of earlier results and examples) and an application on knots (explained in full detail). [Ry] of H. Ryder is another paper written by a geometer attempting to develop an algebraic theory of LDLQ's *(it is almost unreadable, because it does not use algebraic notation and terminology, but it might contain interesting algebraic results.)*. The paper [Bri] of E. Brieskorn *(which I have never seen)* seems to be of a similar kind.

Group conjugation is a fundamental example: for a group G, put $a \circ b = aba^{-1}$. Then $G(\circ)$ is an LDI left quasigroup. Joyce [Jo1] proves that, in the language of multiplication and left division, the equational theory of group conjugation is precisely that of LDI left quasigroups. The same is true also in the language of multiplication, see for instance [St2].

Joyce's paper. Probably the most important paper on LDI left quasigroups (called quandles there) is [Jo1] of D. Joyce: he invents there the knot quandle (see next section). Quandles are considered with the operation of multiplication and left division.

As already stated, the equational theory of group conjugation coincides with that of quandles. For medial quandles (called *abelian* by Joyce), the free medial quandle over X is the subquandle of $\mathrm{Conj}(G)$ generated by X, where G is a group presented by $\langle X;\ ab^{-1}c = cb^{-1}a,\ a,b,c \in X\rangle$. The free n-generated medial involutory quandle is a subquandle of the core of $\mathbb{Z}^{n-1}$ generated by $(0,\dots,0)$, $(1,0,\dots,0),\dots,(0,\dots,0,1)$; hence the variety of LSMI groupoids is generated by cores abelian groups. The 2-generated free involutory quandle is isomorphic to the core of $\mathbb{Z}$.

An *augmented* quandle is a quandle with an action of a group added. This notion is used in the knot theoretical part heavily. Quandles are representable by conjugation classes and by coset classes. Involutory quandles are representable by sets covered by well-behaving families of mappings from the core of $\mathbb{Z}$ (so called *involutory quandles with geodesics*).

Nobusawa's later papers. LDI left quasigroups were studied in N. Nobusawa's later papers (he called them pseudo-symmetric sets). [No4] contains basic observations and some properties of simple LDI LQ's. If G is the subgroup of the symmetric group over A generated by all left translations in a simple LDI LQ A, then the commutator G' is a minimal normal subgroup of G (cf. with simplicity of LSLDI groupoids in [Na]). Moreover, the converse is true, provided A is left ideal-free and all left translations in A are pairwise different. An application to simplicity of groups (via simplicity of their conjugation LDI left quasigroup) is shown.

Some structure theory of LDI LQ's is developed in [No6]. Again, the group of displacements plays an important role. A close connection between nilpotency and solvability of an LDI LQ and its group of displacements is shown. The paper [No7] deals with LDI left quasigroups which are not a subgroupoid of a conjugation LDI left quasigroup. Finally, [No8] presents an analogy of the group-theoretical Jordan-Hölder theorem for LDI left quasigroups.

7. The knot quandle

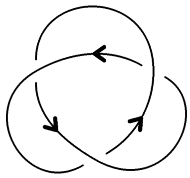
7.1. **The knot quandle.** The main reason that led to the development of the theory of LDI left quasigroups was the discovery of the so called *knot quandle* (independently by D. Joyce [Jo1] and S. V. Matveev [Ma] in 80's), an LDI left quasigroup assigned to a knot, which turns out to be a full knot invariant. The construction works as follows.

A *knot* is a subspace of a (3-dimensional) sphere S^3 homeomorphic to a circle. Two knots K_1 and K_2 are *equivalent*, if there is an (auto)homeomorphism of the sphere such that its restriction to K_1 is a homeomorphism of K_1 and K_2. One of the most important problems in the knot theory is, given two knots, to decide, whether they are equivalent. For this reason, invariants (with respect to the equivalence) of knots are searched.

Usually only *tame* knots are studied (tame means equivalent to a close polygonal curve). A classical invariant of tame knots is the fundamental group. It is defined for a knot K to be the fundamental group of the topological space $S^3 \setminus K$. It is a full invariant, i.e. two tame knots are equivalent, iff their fundamental groups are isomorphic. The problem is that fundamental groups are usually not very well computable, so one would like to look for "more simple" invariants. The knot quandle is a possibility.

Consider a *regular projection* of a tame knot, i.e. a mapping to a (2-dimensional) plane such that there are only finitely many crossings and none of them is a three-fold crossing. This projection divides the knot to arcs (by an *arc* we mean a segment from one underpass through some overpasses to the next underpass); denote the set of arcs A. Choose an orientation of the knot and define the *knot quandle* to be an LDI left quasigroup generated by A and presented by relations (1) $ab = c$ for every underpass, where "a coming under b yields c" and where b is going over a in the left-right direction (in a sense of the orientation of the knot), and (2) $a\backslash b = c$ in the situation, where b is going over a in the right-left direction. The *involutory knot quandle* is an LSLDI groupoid (i.e. an LDI left quasigroup satisfying $x\backslash y \approx xy$) defined analogously regardless the orientation. Examples:

(i) a



trefoil

The trefoil quandle is presented by

$$\langle a, b, c;\ ab = c,\ bc = a,\ ca = b\rangle\,.$$

One can easily compute that the involutory trefoil quandle is the core of the cyclic group $\mathbb{Z}_3$. (The core of an abelian group G is the groupoid $G(*)$ defined by $a * b = 2a - b$.)

(ii)

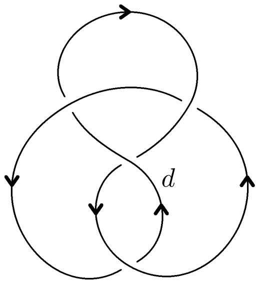


figure-eight

The the figure-eight quandle is presented by

$$\langle a, b, c, d;\ ab = d,\ b\backslash c = a,\ cd = b,\ d\backslash a = c\rangle\,.$$

One can easily compute that the involutory figure-eight quandle is the core of the cyclic group $\mathbb{Z}_5$ (recall that it is *generated* by arcs, not the set of arcs).

Joyce and Matveev proved that two tame knots are equivalent, iff their knot quandles are isomorphic. (Thus, in particular, the knot quandle does not depend on the chosen regular projection.) The proof is based on investigation of the role of conjugation in fundamental groups and it is rather difficult. Involutory knot quandles are also invariant, however, there are two non-equivalent tame knots with isomorphic involutory knot quandles; on the other hand, known examples are rather complicated.

Further knot applications. Involutory quandles appear in several papers about knots. For instance, H. Azcan and R. Fenn [AF] study involutory quandles with geodesics (called spherical quandles there) and their application to knot theory. The paper [Ka] of L. H. Kauffman is devoted to the foundations of the virtual knot theory and the (involutory) virtual knot quandle is defined there.

8. Non-idempotent LSLD groupoids

It seemed formerly that non-idempotent selfdistributive structures (whether symmetric or not) do not play any significant role. This was true for two-sided (i.e.

both left and right) distributivity, but in last years, one-sided distributive non-idempotent systems arised in various situations. We refer to the book [De] of P. Dehornoy.

The first paper about non-idempotent left symmetric left distributive groupoids seems to be [Ke2] of Tomáš Kepka. He states basic structural properties and makes observations about the lattice of subvarieties. His investigations have been continued recently with E. Jeřábek and D. Stanovský in [JKS], where a description of subdirectly irreducible non-idempotent LSLD groupoids is provided. The note [St1] discusses a couple of LSLD operations on groups. Normal forms of terms in LSLD, LSM, LSLDI and LSMI are found there.

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LSLD

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