Abelianess and Solvability

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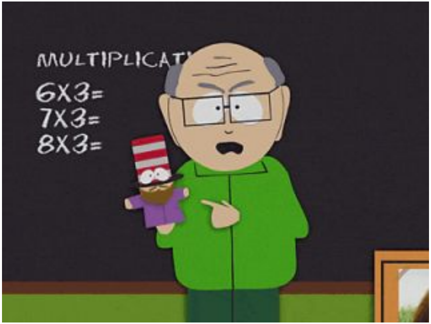
What is a solvable algebra?



Theorem (Galois 1830s)

Gal(f) is solvable (as a group) iff f is solvable (in radicals).

In groups

A group G is *solvable* iff

- there are $N_i \trianglelefteq G$ such that $1 = N_0 \leq N_1 \leq \dots \leq N_k = G$ and N_{i+1}/N_i are abelian groups
- there are abelian groups A_i s.t. $G \simeq A_1 : (A_2 : \dots (A_{k-1} : A_k))$

A group G is *nilpotent* iff

- there are $N_i \trianglelefteq G$ such that $1 = N_0 \leq N_1 \leq \dots \leq N_k = G$ and $N_{i+1}/N_i \leq Z(G/N_i)$
- there are abelian groups A_i s.t. $G \simeq A_1 :_c (A_2 :_c \dots (A_{k-1} :_c A_k))$

$G = A : F$ is an *(abelian) extension*: $A \trianglelefteq G$ is abelian, $G/A \simeq F$ and

$$(a, x)(b, y) = (\varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y}, xy), \qquad \varphi_{x,y}, \psi_{x,y} \in \mathrm{Aut}(A)$$

... *central extension* iff $\varphi = \psi = id$

In finite groups

Structural theorems, e.g.,

- p-groups are nilpotent
- finite nilpotent groups are direct products of p-groups
- (Feit, Thompson) groups of odd order are solvable

Characterizations, e.g.,

- Galois' theorem
- a finite group is solvable iff it is not *Boolean complete*

BC = there is a polynomial subreduct isomorphic to 2-elt Boolean a.

Computational problems, e.g.,

- *equation solving:* nilpotent $\Rightarrow$ P, not solvable $\Rightarrow$ NP-complete
- *identity checking:* nilpotent $\Rightarrow$ P, not solvable $\Rightarrow$ coNP-complete
- *circuit evaluation:* solvable $\Rightarrow$ ACC^1, not solvable $\Rightarrow$ P-complete

In universal algebra

A is *solvable* if there are $\alpha_i \in \mathrm{Con}(A)$ such that
$0_A = \alpha_0 \leq \alpha_1 \leq ... \leq \alpha_k = 1_A$ and α_{i+1}/α_i is an abelian congr. in A/α_i

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A is *abelian* if dtto with $k = 1$

α abelian in A iff (TC) for every term t and every $x \,\alpha\, y$, $u_i \,\alpha\, v_i$

$\alpha \in \zeta(A)$ iff (TC) for every term t and every $x \,\alpha\, y$ and u_i, v_i arbitrary

(TC)
$t(x, u_1, \ldots, u_n) = t(x, v_1, \ldots, v_n) \;\Rightarrow\; t(y, u_1, \ldots, u_n) = t(y, v_1, \ldots, v_n)$

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Abelian extensions?
Other nice properties?

TCT: type 3 $\Rightarrow$ BC; no type 3 $\Rightarrow$???

In algebras with a Mal'tsev term

$m(x, x, y) = m(y, x, x) = y$

Fact (Gumm-Smith):
An algebra with a Mal'tsev term is *abelian* iff it is poly. equiv. to a module.

But abelian congruences / extensions are less clear.

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So are generalizations of many group theory results. Nevertheless:

TCT: no type 3 $\Rightarrow$ solvable $\Leftrightarrow$ no 2-snags $\Rightarrow$ not BC
Hence, "solvable iff not BC" for algebras with a Mal'tsev term

(Horváth) identity checking, equation solving as in groups

In loops

A *loop* is an algebra $(L, \cdot, \backslash, 1)$ such that

- $1x = x1 = x$
- $\forall\ x, y$ there are unique $u = x\backslash y$, $v = y/x$ such that $xu = y$, $vx = y$

Mal'tsev term: $m(x, y, z) = (x/y)z$

$\mathrm{Mlt}(L) = \langle L_a, R_a : a \in L\rangle$
$\mathrm{Inn}(L) = \mathrm{Mlt}(L)_1 = \langle L_{a,b}, R_{a,b}, T_a : a, b \in L\rangle$

$L_{a,b} = L_{ab}^{-1} L_a L_b$
$L_{a,b} = R_{ba}^{-1} R_a R_b$
$T_a = L_a^{-1} R_a$

Normal subloops $\leftrightarrow$ congruences

$=$ kernels of a homomorphisms

$=$ subloops invariant with respect to $\mathrm{Inn}(L)$

In loops

Fact: A loop is *abelian* iff it is an abelian group

Fact: $Z(L) = \{a \in L : ax = xa, a(xy) = (ax)y, (xa)y = x(ay)\ \forall\ x, y \in L\}$

Hence L is *nilpotent* if there are $N_i \trianglelefteq L$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and $N_{i+1}/N_i \leq Z(L/N_i)$

Fact: TFAE

- $A \leq Z(L)$
- $\varphi_{r,s}(a) = 1$ for every $a \in A$, $r, s \in L$, $\varphi \in \{L, R, T\}$
- $L \simeq A :_c F$ where A is an abelian group, with operation

$$(a, x)(b, y) = (a + b + \theta_{x,y}, xy)$$

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!!! $A \trianglelefteq L$ is an abelian group $\not\Rightarrow$ the corresp. congr. is abelian in L !!!

In loops

Main Theorem (S., Vojtěchovský)

L loop, $A \trianglelefteq L$ then the following are equivalent:

- *A is abelian in L*
- *$\varphi_{r,s}(a) = \varphi_{u,v}(a)$ for every $a, r/u, s/v \in A$, $\varphi \in \{L, R, T\}$*
- *$L \simeq A : F$ where A is an abelian group, with operation*

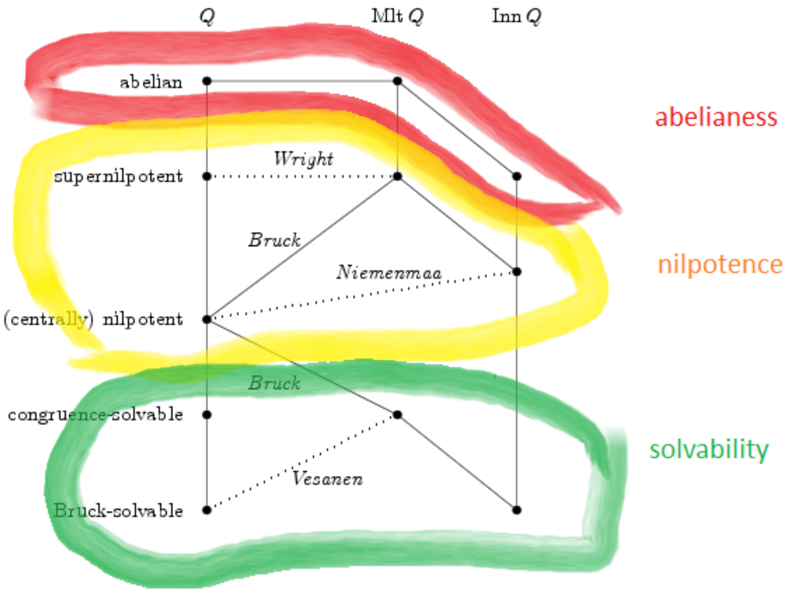
$$(a,x)(b,y) = (\varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y}, xy)$$

  where $\varphi_{x,y}, \psi_{x,y} \in \mathrm{Aut}(A)$, $\theta_{x,y} \in A$ with $\varphi_{x,1} = \psi_{1,x} = 1$, $\theta_{x,1} = \theta_{1,x} = 0$.

Compare:

- $A \leq Z(L)$
- $\varphi_{r,s}(a) = a$ for every $a \in A$, $r, s \in L$, $\varphi \in \{L, R, T\}$
- $L \simeq A :_c F$ with $(a,x)(b,y) = (a + b + \theta_{x,y}, xy)$

Loops and their associated groups



Feit-Thompson theorem

Theorem (Feit-Thompson 1962)

Groups of odd order are solvable.

Theorem (Glauberman 1964/68)

Moufang loops of odd order are weakly solvable.

L is *weakly solvable* if there are $H_i \leq L$ such that
$1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = L$ and H_{i+1}/H_i are abelian groups

Problem

Are Moufang loops of odd order solvable?
Are other loops of ??? ????? solvable?


WOW, WHAT A GREAT
AUDIENCE!