Commutator Theory for Loops

David Stanovský

Charles University, Prague, Czech Republic
stanovsk@karlin.mff.cuni.cz

joint work with Petr Vojtěchovský, University of Denver

June 2013

Feit-Thompson theorem

Theorem (Feit-Thompson, 1962)

Groups of odd order are solvable.

Can be extended?

- To which *class of algebras* ? (containing groups)
- What is *odd order* ?
- What is *solvable* ?

Feit-Thompson theorem

Theorem (Feit-Thompson, 1962)

Groups of odd order are solvable.

Can be extended?

- To which *class of algebras* ? (containing groups)
- What is *odd order* ?
- What is *solvable* ?

Theorem (Glauberman 1964/68)

Moufang loops of odd order are solvable.

Moufang loop = replace associativity by $x(z(yz)) = ((xz)y)z$

solvable = there are $N_i \trianglelefteq L$ such that $1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and N_{i+1}/N_i are abelian groups.

Loops

A *loop* is an algebra $(L, \cdot, 1)$ such that

- $1x = x1 = x$
- for every x, y there are unique u, v such that $xu = y$, $vx = y$

For universal algebra purposes: $(L, \cdot, \backslash, /, 1)$, where $u = x\backslash y$, $v = y/x$.

Examples:

- octonions $\rightsquigarrow$ Moufang loops
- various other classes of weakly associative loops
- various combinatorial constructions, e.g., from Steiner triples systems, coordinatization of projective geometries, etc.

Loops

A *loop* is an algebra $(L, \cdot, 1)$ such that

- $1x = x1 = x$
- for every x, y there are unique u, v such that $xu = y$, $vx = y$

For universal algebra purposes: $(L, \cdot, \backslash, /, 1)$, where $u = x\backslash y$, $v = y/x$.

Examples:

- octonions $\rightsquigarrow$ Moufang loops
- various other classes of weakly associative loops
- various combinatorial constructions, e.g., from Steiner triples systems, coordinatization of projective geometries, etc.

Normal subloops $\leftrightarrow$ congruences

$=$ kernels of a homomorphisms

$=$ subloops invariant with respect to $\mathrm{Inn}(L)$

$$\mathrm{Inn}(L) = \mathrm{Mlt}(L)_1, \qquad \mathrm{Mlt}(L) = \langle L_a, R_a : a \in L \rangle$$

Solvability, nilpotence - after R. H. Bruck

Bruck's approach (1950's), by direct analogy to group theory:

L is *solvable* if there are $N_i \trianglelefteq L$ such that $1 = N_0 \leq N_1 \leq ... \leq N_k = L$ and N_{i+1}/N_i are abelian groups.

L is *nilpotent* if there are $N_i \trianglelefteq L$ such that $1 = N_0 \leq N_1 \leq ... \leq N_k = L$ and $N_{i+1}/N_i \leq Z(L/N_i)$.

$Z(L) = \{a \in L : ax = xa, a(xy) = (ax)y, (xa)y = x(ay) \ \forall \ x, y \in L\}$

Solvability, nilpotence - after R. H. Bruck

Bruck's approach (1950's), by direct analogy to group theory:

L is *solvable* if there are $N_i \trianglelefteq L$ such that $1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and N_{i+1}/N_i are abelian groups.

L is *nilpotent* if there are $N_i \trianglelefteq L$ such that $1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and $N_{i+1}/N_i \leq Z(L/N_i)$.

$Z(L) = \{a \in L : ax = xa, a(xy) = (ax)y, (xa)y = x(ay)\ \forall\ x, y \in L\}$

Alternatively, define

$$L^{(0)} = L_{(0)} = L, \qquad L_{(i+1)} = [L_{(i)}, L_{(i)}], \qquad L^{(i+1)} = [L^{(i)}, L]$$

- L *solvable* iff $L_{(n)} = 1$ for some n
- L *nilpotent* iff $L^{(n)} = 1$ for some n

Need commutator!

- $[N, N]$ is the smallest M such that N/M is abelian
- $[N, L]$ is the smallest M such that $N/M \leq Z(L/M)$
- $[N_1, N_2]$ is ???

Solvability, nilpotence - universal algebra way

Commutator theory approach (1970's):

L is *solvable* if there are $\alpha_i \in \mathrm{Con}(L)$ such that
$0_L = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_L$ and α_{i+1} is abelian over α_i.

L is *nilpotent* if there are $\alpha_i \in \mathrm{Con}(L)$ such that
$0_L = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_L$ and $\alpha_{i+1}/\alpha_i \leq \zeta(L/\alpha_i)$.

$\zeta(L) =$ the largest ζ such that $C(\zeta, 1_L; 0_L)$, i.e., $[\zeta, 1_L] = 0_L$.

Alternatively, define

$$\alpha^{(0)} = \alpha_{(0)} = 1_L, \qquad \alpha_{(i+1)} = [\alpha_{(i)}, \alpha_{(i)}], \qquad \alpha^{(i+1)} = [\alpha^{(i)}, 1_L]$$

- L *solvable* iff $\alpha_{(n)} = 1$ for some n
- L *nilpotent* iff $\alpha^{(n)} = 1$ for some n

We have a commutator!

- $[\alpha, \alpha]$ is the smallest β such that α/β is abelian
- $[\alpha, 1_L]$ is the smallest β such that $\alpha/\beta \leq \zeta(L/\beta)$
- $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$

Translating to loops I

Good news

1. *A loop is abelian if and only if it is an abelian group.*
2. *The congruence center corresponds to the Bruck's center.*

Hence, nilpotent loops are really nilpotent loops!

Translating to loops I

Good news

1. *A loop is abelian if and only if it is an abelian group.*
2. *The congruence center corresponds to the Bruck's center.*

Hence, nilpotent loops are really nilpotent loops!

Nevertheless, supernilpotence is a stronger property.

Translating to loops II

Bad news

Abelian congruences $\neq$ normal subloops that are abelian groups

N is an abelian group iff $[N,N]_N = 0$, i.e., $[1_N, 1_N]_N = 0_N$

N is *abelian in L* iff $[N,N]_L = 0$, i.e., $[\nu,\nu]_L = 0_L$

abelian $\neq$ abelian in L !!!

Example: $L = \mathbb{Z}_4 \times \mathbb{Z}_2$, redefine $(a,1)+(b,1) = (a*b,0)$

$*$	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

- $N = \mathbb{Z}_4 \times \{0\} \trianglelefteq L$
- N is an abelian group
- $[N,N]_L = N$, hence N is not abelian in L

Two notions of solvability

L is *Bruck-solvable* if there are $N_i \trianglelefteq L$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and N_{i+1}/N_i are *abelian groups*
(i.e. $[N_{i+1}, N_{i+1}]_{N_{i+1}} \leq N_i$)

L is *congruence-solvable* if there are $N_i \trianglelefteq L$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = L$ and N_{i+1}/N_i are *abelian in L/N_i*
(i.e. $[N_{i+1}, N_{i+1}]_L \leq N_i$)

The loop from the previous slide is

- Bruck-solvable
- NOT congruence-solvable

Commutator in loops

$L_{a,b} = L_{ab}^{-1} L_a L_b, \qquad R_{a,b} = R_{ba}^{-1} R_a R_b, \qquad T_a = L_a R_a^{-1}$
$M_{a,b} = M_{b\backslash a}^{-1} M_a M_b, \qquad U_a = R_a^{-1} M_a$

$\mathrm{TotMlt}(L) = \langle L_a, R_a, M_a : a \in L \rangle$
$\mathrm{TotInn}(L) = \mathrm{TotMlt}(L)_1 = \langle L_{a,b}, R_{a,b}, T_a, M_{a,b}, U_a : a, b \in L \rangle$

Main Theorem

$\mathcal{V}$ a variety of loops, Φ a set of words that generates TotInn's in $\mathcal{V}$, then

$$[\alpha, \beta] = Cg((\varphi_{u_1,\dots,u_n}(a), \varphi_{v_1,\dots,v_n}(a)) : \ \varphi \in \Phi, \ 1 \, \alpha \, a, \ u_i \, \beta \, v_i)$$

for every $L \in \mathcal{V}$, $\alpha, \beta \in \mathrm{Con}(L)$.

Examples:

- in loops, $\Phi = \{L_{a,b}, R_{a,b}, M_{a,b}, T_a, U_a\}$
- in groups, $\Phi = \{T_a\}$

Commutator in loops

$$L_{a,b} = L_{ab}^{-1} L_a L_b, \qquad R_{a,b} = R_{ba}^{-1} R_a R_b, \qquad T_a = L_a R_a^{-1}$$
$$M_{a,b} = M_{b\backslash a}^{-1} M_a M_b, \qquad U_a = R_a^{-1} M_a$$

$$\mathrm{TotMlt}(L) = \langle L_a, R_a, M_a : a \in L \rangle$$
$$\mathrm{TotInn}(L) = \mathrm{TotMlt}(L)_1 = \langle L_{a,b}, R_{a,b}, T_a, M_{a,b}, U_a : a, b \in L \rangle$$

Main Theorem

$\mathcal{V}$ a variety of loops, Φ a set of words that generates TotInn's in $\mathcal{V}$, then

$$[\alpha, \beta] = Cg((\varphi_{u_1,\dots,u_n}(a), \varphi_{v_1,\dots,v_n}(a)) : \ \varphi \in \Phi, \ 1 \, \alpha \, a, \ u_i \, \beta \, v_i)$$

for every $L \in \mathcal{V}$, $\alpha, \beta \in \mathrm{Con}(L)$.

Examples:

- in loops, $\Phi = \{L_{a,b}, R_{a,b}, M_{a,b}, T_a, U_a\}$
- in groups, $\Phi = \{T_a\}$

$$[M, N]_L = Ng(\varphi_{u_1,\dots,u_n}(a) / \varphi_{v_1,\dots,v_n}(a) : \ \varphi \in \Phi, \ a \in M, \ u_i / v_i \in N)$$

Solvability and nilpotence summarized

$$\mathrm{Mlt}(L) \text{ nilpotent}$$
$$\Downarrow \text{(trivial)}$$
$$\mathrm{Inn}(L) \text{ nilpotent}$$
$$\Downarrow \text{(Niemenmaa)}$$
$$L \text{ nilpotent}$$
$$\Downarrow \text{(Bruck)}$$
$$\mathrm{Mlt}(L) \text{ solvable}$$
$$\Downarrow \text{(Vesanen)}$$
$$L \text{ Bruck-solvable}$$

Where to put "L congruence-solvable" ?

Solvability and nilpotence summarized

$$\mathrm{Mlt}(L) \text{ nilpotent}$$

$$\Downarrow \text{ (trivial)}$$

$$\mathrm{Inn}(L) \text{ nilpotent}$$

$$\Downarrow \text{ (Niemenmaa)}$$

$$L \text{ nilpotent}$$

$$\Downarrow \text{ (Bruck)}$$

$$\mathrm{Mlt}(L) \text{ solvable}$$

$$\Downarrow \text{ (Vesanen)}$$

$$L \text{ Bruck-solvable}$$

Where to put "L congruence-solvable" ?

We know: stronger Vesanen fails.

Problem

Let L be a congruence-solvable loop. Is the group $\mathrm{Mlt}(L)$ solvable?

Feit-Thompson revisited

Theorem (Glauberman 1964/68)

Moufang loops of odd order are Bruck-solvable.

Problem

Are Moufang loops of odd order congruence-solvable?

For Moufang loops,

- we know that abelian $\neq$ abelian in L (in a 16-element loop)
- is it so that Bruck-solvable iff congruence-solvable?