Outline

1. Motivation: Where self-distributivity comes from

2. Medial, trimedial and distributive quasigroups
   2a. affine representation in general
   2b. representation of distributive and trimedial quasigroups over commutative Moufang loops
   2c. the structure of distributive quasigroups

3. Left distributive quasigroups (and quandles)
   3a. (almost) linear representation
   3b. homogeneous representation

4. Applications in knot theory
1a. Historical motivation
Let \((A, \ast)\) be a binary algebraic structure.

**Left translations** are mappings \(L_a : A \to A, x \mapsto a \ast x\).

\((A, \ast)\) is left self-distributive if all \(L_a\)’s are endomorphisms.

\[ a \ast (x \ast y) = (a \ast x) \ast (a \ast y) \]

"I think that there is a philosophical difference between an associative world and a distributive world. The associative world is a geometric world; a world in which space and time are important and fundamental concepts. The distributive world seems different to me. I think that it is a quantum world without space and time, in which only information exists."

Dan Moskovich
Self-distributivity

Let \((A, \ast)\) be a binary algebraic structure.

*Left translations* are mappings \(L_a : A \to A, \; x \mapsto a \ast x\).

\((A, \ast)\) is *left self-distributive* if all \(L_a\)'s are endomorphisms.

\[ a \ast (x \ast y) = (a \ast x) \ast (a \ast y) \]

Self-distributivity appears naturally in

- low dimensional topology (knot and braid invariants)
- set theory (Laver’s groupoids of elementary embeddings)
- Loos’s symmetric spaces
- etc.
Self-distributive quasigroups

\((Q, \ast)\) is a quasigroup if \(a \ast x = b,\ y \ast a = b\) have unique solutions \(\forall a, b\)

I.e., all left and right translations are permutations

In combinatorics: latin squares = (finite) quasigroups

Early studies on self-distributive quasigroups:
- Burstin, Mayer: *Distributive Gruppen von endlicher Ordnung* (1929)
- Anton Sushkevich: Lagrange’s theorem under weaker assumptions
- Toyoda, Murdoch, Bruck: medial quasigroups are affine (1940s)
- Orin Frink: abstract definition of mean value (1950s)
- Sherman Stein (1950s)
- Soviet school: V. D. Belousov, V. M. Galkin, V. I. Onoi (1960-70s)
Spaces with reflection

In a space $X$ (euclidean or wherever it makes sense), let

$$a \ast b = \text{the reflection of } b \text{ over } a.$$ 

Then $(X, \ast)$ is

- left distributive
- idempotent
- $a \ast x = b$ always has a unique solution, $x = a \ast b$
- (usually not a quasigroup, e.g. on a sphere)

Nowadays we say $(X, \ast)$ is an involutory quandle.

- observed by Takasaki (1942)
- elaborated by Loos (1960s): symmetric spaces
Conjugation in groups

In a group $G$, let $a \star b = aba^{-1}$.

Then $(G, \star)$ is
- left distributive
- idempotent
- $a \star x = b$ always has a unique solution, $x = a^{-1}ba$

Nowadays we say $(G, \star)$ is a **quandle**.

**Observation**: Left distributive quasigroups are quandles.

- Stein (1959): left distributive quasigroups embed into conjugation quandles (quandles do not, in general)
- Conway and Wraith (1960s): *wrack of a group*
- Joyce and Matveev (1982): quandles as knot invariants
Knot coloring: 3-coloring

To every arc, assign one of three colors in a way that every crossing has one or three colors.

Invariant: count non-trivial (non-monochromatic) colorings.
Knot coloring: Fox $n$-coloring

To every arc, assign one of $n$ colors, 0, ..., $n - 1$, in a way that

at every crossing, $2 \cdot \text{bridge} = \text{left} + \text{right}$, modulo $n$

Invariant: count non-trivial colorings.
Quandle coloring

Fix a set $C$ of colors, and a ternary relation $T$ on $C$.
To every arc, assign one of the colors in a way that

$$(c(\alpha), c(\beta), c(\gamma)) \in T$$

**Invariant:** count non-trivial colorings. Really?
Quandle coloring

Fix a set $C$ of colors, and a ternary relation $T$ on $C$. To every arc, assign one of the colors in a way that

$$(c(\alpha), c(\beta), c(\gamma)) \in T$$

Invariant: count non-trivial colorings. Really?

Fact (implicitly Joyce, Matveev (1982))

*Coloring by $(C, T)$ is an invariant if and only if $T$ is a graph of a quandle.*
Quandle coloring

Fact (implicitly Joyce, Matveev (1982))

Coloring by \((C, T)\) is an invariant if and only if \(T\) is a graph of a quandle.

\[ a \ast a = a \]
unique left division

\[ a \ast (b \ast c) = (a \ast b) \ast (a \ast c) \]
2a. Linear and affine representation of quasigroups
Quasigroups and loops

If I say “quasigroup \((Q, \ast)\)“, I often implicitly mean \((Q, \ast, \backslash, /)\).

If I say “loop \((Q, \cdot)\)“, I often implicitly mean \((Q, \cdot, \backslash, /, 1)\).

(For universal algebraic considerations, you need \(\backslash, /\) as basic operations.)
Term / polynomial equivalence

*term operation* = any composition of basic operations

*polynomial operation* = term op. with some var’s substituted by constants

two algebras are *term equivalent* if they have the same term operations

two algebras are *poly. equivalent* if they have the same poly. operations

... “the two algebras are essentially the same algebraic object”

Examples:

- term equivalent: group \((G, \cdot,^{-1}, 1)\) and the corresponding associative loop \((G, \cdot, /, \setminus, 1)\)
- term equivalent: Boolean algebra and the corresponding Boolean ring
- polynomially equivalent: the quasigroup \((\mathbb{Q}, \text{arithmetic mean})\) and the module \(\mathbb{Z}[1/2]\)-module \(\mathbb{Q}\).

Observation:

- term equivalent algebras have identical subalgebras
- polynomially equivalent algebras have identical congruences
Loop isotopes

**isotope** = shuffle rows and columns, rename elements in the table

**Fact**

*Given a quasigroup \((Q, \ast)\), the only loop isotopes (up to isomorphism) are \((Q, \cdot)\) with \(a \cdot b = (a/e_1) \ast (e_2/b)\), with \(e_1, e_2 \in Q\) arbitrary.*

**Note:** The loop operation \(\cdot\) is polynomial over \((Q, \ast)\).

We can recover the quasigroup operation as \(a \ast b = R_{e_1}(a) \cdot L_{e_2}(b)\).

- this is rarely a polynomial operation over \((Q, \cdot)\)
- the best case: \(\ast\) is a linear / affine form over \((Q, \cdot)\)
  - i.e. \(R_{e_1}, L_{e_2}\) are linear / affine mappings over \((Q, \cdot)\)
Linear / affine quasigroups

A permutation \( \varphi \) of \( Q \) is \textit{affine} over \((Q, \cdot)\) if

\[
\varphi(x) = \tilde{\varphi}(x) \cdot u \quad \text{or} \quad \varphi(x) = u \cdot \tilde{\varphi}(x)
\]

where \( \tilde{\varphi} \) is an automorphism of \((Q, \cdot)\) and \( u \in Q \).

\textit{Affine quasigroup} over a loop \((Q, \cdot)\) is \((Q, \ast)\) with

\[
a \ast b = \varphi(a) \cdot \psi(b)
\]

for some affine mappings \( \varphi, \psi \) over \((Q, \cdot)\) such that \( \tilde{\varphi} \tilde{\psi} = \tilde{\psi} \tilde{\varphi} \).
Linear / affine quasigroups

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\[
a \ast b = \varphi(a) \cdot \psi(b)
\]

for some affine mappings \( \varphi, \psi \) over \((Q, \cdot)\) such that \( \tilde{\varphi} \tilde{\psi} = \tilde{\psi} \tilde{\varphi} \).

\textit{Linear quasigroups} over a loop \((Q, \cdot)\): \( \varphi, \psi \) are automorphisms, i.e. \( u = v = 1 \).

Example:

- the quasigroup \((\mathbb{Q}, \text{arithmetic mean})\) is linear over the group \((\mathbb{Q}, +)\)
- the quasigroup \((\mathbb{O}, \ast)\) with \( x \ast y = ix \cdot jy \) is affine over the octonion loop \((\mathbb{O}, \cdot)\)
- quasigroups affine over abelian groups are medial (see blackboard)
Module-theoretical point of view

How to turn an affine representation into a polynomial equivalence? (remember: the loop isotope is always polynomial over the quasigroup)

Consider wlog \( \varphi(x) = u \cdot \tilde{\varphi}(x) \), \( \psi(x) = v \cdot \tilde{\psi}(x) \).

Then \( x \ast y = \varphi(x) \cdot \psi(y) = (u \cdot \tilde{\varphi}(x)) \cdot (v \cdot \tilde{\psi}(y)) \) is a polynomial operation over the algebra \( (Q, \cdot, \tilde{\varphi}, \tilde{\psi}) \).

Conversely, \( \cdot, \tilde{\varphi}, \tilde{\psi} \) are polynomial operations over \( (Q, \ast) \),

\[ \tilde{\varphi}(x) = (x \ast e_1) / (1 \ast e_1) \]

Hence, \( (Q, \ast) \) and \( (Q, \cdot, \tilde{\varphi}, \tilde{\psi}) \) are polynomially equivalent.
Module-theoretical point of view

$(Q, \ast)$ and $(Q, \cdot, \bar{\varphi}, \bar{\psi})$ are polynomially equivalent.

What is $(Q, \cdot, \bar{\varphi}, \bar{\psi})$, a loop expanded by commuting automorphisms?

The classical case: the loop is an abelian group, $(Q, +)$.

Then $(Q, +, \bar{\varphi}, \bar{\psi})$ is term equivalent to a module over Laurent polynomials $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$:

- the additive structure is $(Q, +)$
- the action of $s, t$ is that of $\bar{\varphi}, \bar{\psi}$, respectively

The corresponding quasigroup operation can be written as an affine form:

$$x \ast y = sx + ty + c.$$
Module-theoretical point of view

$(Q, \ast)$ and $(Q, \cdot, \tilde{\varphi}, \tilde{\psi})$ are polynomially equivalent.

What is $(Q, \cdot, \tilde{\varphi}, \tilde{\psi})$, a loop expanded by commuting automorphisms?

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- the action of $s, t$ is that of $\tilde{\varphi}, \tilde{\psi}$, respectively

The corresponding quasigroup operation can be written as an affine form:

\[ x \ast y = sx + ty + c. \]

General case: The same idea works, forget associativity of $(Q, +)$.

Loops expanded by automorphisms = “non-associative modules”

(things work particularly nicely e.g. for diassociative loops)
An outline of the representation theorems

\begin{center}
\begin{tikzpicture}
\node (q) {quasigroups};
\node (l) [right of=q] {loops};
\node (md) [below of=q, yshift=-1cm] {medial};
\node (di) [below of=q, yshift=-1cm] {distributive (trimedial)};
\node (ld) [below of=q] {left distributive};
\node (il) [below of=l, yshift=-1cm] {involutory l.d.};
\node (lb) [below of=l] {Belousov-Onoi loops};
\node (ag) [right of=l] {abelian groups};
\node (cm) [right of=ag] {commutative Moufang loops};
\node (bl) [right of=bl] {B-loops};
\node (rl) [right of=ld, xshift=-1cm] {right linear, left quadratic};
\node (rla) [right of=md, xshift=1cm] {linear / affine};
\node (la) [right of=il, xshift=-1cm] {linear (affine)};
\node (rll) [right of=lb, xshift=-1cm] {right linear, left quadratic};
\node (rlaq) [right of=il, xshift=1cm] {right linear, left quadratic};
\path (q) edge (md)
(q) edge (di)
(q) edge (ld)
(l) edge (ag)
(l) edge (cm)
(l) edge (bl)
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\end{tikzpicture}
\end{center}
2b. Medial, trimedial and distributive quasigroups

(Belousov, Soublin, Kepka, 1960s-70s)
Medial quasigroups are affine over abelian groups

**Mediality** = the identity \((x \ast y) \ast (u \ast v) = (x \ast u) \ast (y \ast v)\)

**Note:** medial idempotent quasigroups are left and right distributive

---

**Theorem (Toyoda-Murdoch-Bruck, 1940’s)**

The following are equivalent for a quasigroup \((Q, \ast)\):

1. it is medial,
2. it is affine over an abelian group.

Moreover, for an idempotent quasigroup, TFAE:

1. it is **medial idempotent**,
2. it is **linear** over an abelian group and \(\varphi = 1 - \psi\), i.e.
   \[ x \ast y = (1 - \psi)(x) + \psi(y) = x - \psi(x) + \psi(y). \]
Medial quasigroups are affine over abelian groups

Theorem (Toyoda-Murdoch-Bruck, 1940's)

The following are equivalent for a quasigroup \((Q, \ast)\):

1. it is medial;
2. it is affine over an abelian group.

(2) ⇒ (1) is straightforward.

(1) ⇒ (2): Pick arbitrary \(e_1, e_2 \in Q\), define \(a \cdot b = (a/e_1) \ast (e_2 \setminus b)\). Prove that

- \((Q, \cdot)\) is a medial loop, hence an abelian group
- the mappings \(\varphi(x) = x/e_1\) and \(\psi(x) = e_2 \setminus x\) are affine over \((Q, \cdot)\).
- the mappings \(\tilde{\varphi}, \tilde{\psi}\) commute
Medial quasigroups are affine over abelian groups

Let \((Q, \ast)\) be a medial quasigroup.
We prove that \((Q, \cdot)\) with \(a \cdot b = (a/e_1) \ast (e_2 \backslash b)\) is a medial loop.

First, prove that \((Q, \circ)\) with \(a \circ b = (a/e_1) \ast b\) is medial.

\[
(a \circ b) \circ (c \circ d) = (((a/e_1) \ast b)/e_1) \ast ((c/e_1) \ast d)
= (((a/e_1) \ast b)/(e_1/e_1) \ast e_1) \ast ((c/e_1) \ast d)
= (((a/e_1)/(e_1/e_1)) \ast (b/e_1)) \ast ((c/e_1) \ast d)
\]

Now, interchange \(b/e_1\) and \(c/e_1\) and get equality to \((a \circ c) \circ (b \circ d)\).

Proving that \((Q, \cdot)\) is medial is a dual argument over \((Q, \circ)\).
Trimedia quasigroups are affine over c. Moufang loops

*trimediality* = every 3-generated subquasigroup is medial
= mediality holds upon any substitution in 3 variables

**Theorem (Kepka, 1976)**

The following are equivalent for a quasigroup \((Q, \ast)\):

1. *it is trimedial*;
2. whenever \((a \ast b) \ast (c \ast d) = (a \ast c) \ast (b \ast d)\), *the subquasigroup* \(\langle a, b, c, d \rangle\) *is medial*,
3. *it satisfies, for every* \(a, b, c \in Q\), *the identities*

\[
(c \ast b) \ast (a \ast a) = (c \ast a) \ast (b \ast a),
\]
\[
(a \ast (a \ast a)) \ast (b \ast c) = (a \ast b) \ast ((a \ast a) \ast c),
\]
4. *it is 1-nuclear affine over a commutative Moufang loop*.

*1-nuclear* = \(x \varphi(x) \in N\), \(x \psi(x) \in N\) for every \(x \in Q\)
Distributive quasigroups are linear over c. Moufang loops

Distributive = both left and right distributive

Corollary (Belousov-Soublin, around 1970)

The following are equivalent for an idempotent quasigroup \((Q, \ast)\):

1. it is trimedial,
2. whenever \((a \ast b) \ast (c \ast d) = (a \ast c) \ast (b \ast d)\), the subquasigroup \(<a, b, c, d>\) is medial,
3. it is distributive,
4. it is 1-nuclear linear over a commutative Moufang loop.

1-nuclear = \(x \varphi(x) \in N, x \psi(x) \in N\) for every \(x \in Q\)
Commutative Moufang loops of order 81 (Kepka-Němec 1981):

- consider the groups $G_1 = (\mathbb{Z}_3)^4$ and $G_2 = (\mathbb{Z}_3)^2 \times \mathbb{Z}_9$
- let $e_1, e_2, e_3, e_4$ be the canonical generators
- let $t_1$ be the triaditive mapping over $G_1$ satisfying

  \[
  t_1(e_2, e_3, e_4) = e_1, \quad t_1(e_3, e_2, e_4) = -e_1, \quad t_1(e_i, e_j, e_k) = 0 \text{ otherwise.}
  \]

- let $t_2$ be the triaditive mapping over $G_2$ satisfying

  \[
  t_2(e_1, e_2, e_3) = 3e_3, \quad t_2(e_2, e_1, e_3) = -3e_3, \quad t_2(e_i, e_j, e_k) = 0 \text{ otherwise.}
  \]

- consider the loops $Q_i = (G_i, \cdot)$ with

  \[
x \cdot y = x + y + t_i(x, y, x - y)
  \]

Sample 1-nuclear automorphisms: $x \mapsto x^{-1}, \; x \mapsto x^2$
Distributive quasigroups are linear over c. Moufang loops

Distributive quasigroups of order 81 (Kepka-Němec 1981):

1. \((G_1, \ast)\) with \(x \ast y = x^{-1} \cdot y^{-1}\)
2. \((G_1, \ast)\) with \(x \ast y = \varphi(x) \cdot \psi(y)\) where 
   \(\varphi(x) = (x_2 - x_1)e_1 - x_2 e_2 - x_3 e_3 - x_4 e_4\) and \(\psi = 1 - \varphi\)
3. \((G_2, \ast)\) with \(x \ast y = \sqrt{x} \cdot \sqrt{y}\)
4. \((G_2, \ast)\) with \(x \ast y = x^{-1} \cdot y^2\)
5. \((G_2, \ast)\) with \(x \ast y = x^2 \cdot y^{-1}\)
6. \((G_2, \ast)\) with \(x \ast y = \varphi(x) \cdot \psi(y)\) where 
   \(\varphi(x) = -x_1 e_1 - x_2 e_2 - (3x_1 + x_3) e_3\) and \(\psi = 1 - \varphi\)

Recall:

- \(G_1 = (\mathbb{Z}_3)^4\) and \(G_2 = (\mathbb{Z}_3)^2 \times \mathbb{Z}_9\)
- \(Q_i = (G_i, \cdot)\) with \(x \cdot y = x + y + t_i(x, y, x - y)\)
Theorem (Kepka, 1976)

The following are equivalent for a quasigroup \((Q, \star)\):

1. it is trimedial,
2. whenever \((a \star b) \star (c \star d) = (a \star c) \star (b \star d)\), the subquasigroup \(\langle a, b, c, d \rangle\) is medial,
3. it satisfies the identities .....,
4. it is 1-nuclear affine over a commutative Moufang loop.

(2) \(\Rightarrow\) (1): \((b \star a) \star (a \star c) = (b \star a) \star (a \star c)\), hence \(\langle a, b, c \rangle\) medial.

(1) \(\Rightarrow\) (3) is obvious.

(3) \(\Rightarrow\) (4). Pick an arbitrary square \(e \in Q\) and define the loop operation on \(Q\) by \(a \cdot b = (a/e) \star (e \backslash b)\). Use a neat theorem of Pflugfelder to prove that this a commutative Moufang loop (plus the other facts).

(4) \(\Rightarrow\) (2). Find a subloop \(Q'\) of \((Q, \cdot)\) that contains all four elements \(a, b, c, d\) and is generated by three elements \(u, v, w\) that associate. Then, by Moufang’s theorem, \(Q'\) is an abelian group, hence \(\langle a, b, c, d \rangle\) medial.
Pflugfelder’s characterization of c. Moufang loops

Theorem (Bruck $1 \iff 2 \iff 3$, Pflugfelder $\iff 4$)

The following are equivalent for a commutative loop $(Q, \cdot)$:

1. it is diassociative and automorphic,
2. it is Moufang,
3. the identity $xx \cdot yz = xy \cdot xz$ holds,
4. the identity $f(x)x \cdot yz = f(x)y \cdot xz$ holds for some $f : Q \to Q$.

Moreover, if $(Q, \cdot)$ is a commutative Moufang loop, than the identity $f(x)x \cdot yz = f(x)y \cdot xz$ holds if and only if $f$ is a $(-1)$-nuclear mapping.

$(-1)$-nuclear $= x^{-1} \varphi(x) \in N$, $x^{-1} \psi(x) \in N$ for every $x \in Q$
2c. The structure of distributive quasigroups
Recap

Distributive quasigroups are essentially the same objects as

- commutative Moufang loops with a 1-nuclear automorphism
- “1-nuclear commutative Moufang modules” over the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$

Idea: use known properties of commutative Moufang loops to reason about distributive quasigroups
Theorem (Fischer-Smith)

Let $Q$ be a finite distributive quasigroup of order $p_1^{k_1} \cdot \ldots \cdot p_n^{k_n}$. Then

$$Q \cong Q_1 \times \ldots \times Q_k$$

where $|Q_i| = p_i^{k_i}$. Moreover, if $Q_i$ is not medial, then $p_i = 3$ and $k_i \geq 4$.

... an analogy holds for commutative Moufang loops
Enumeration

$MI(n) = \text{the number of medial idempotent quasigroups of order } n \text{ up to isomorphism}$

$D(n) = \text{the number of distributive quasigroups of order } n \text{ up to isomorphism}$

**Fisher-Smith says:** with $p_i \neq 3$ pairwise different,

$$D(3^k \cdot p_1^{k_1} \cdot \ldots \cdot p_n^{k_n}) = D(3^k) \cdot MI(p_1^{k_1}) \cdot \ldots \cdot MI(p_n^{k_n})$$

Moreover, $D(3^k) = MI(3^k)$ for $k < 4$. 
Enumeration

Let $Q(Q, \psi)$ denote the quasigroup $(Q, *)$ with $x * y = (1 - \psi)(x) + \psi(y)$.

Observe:
- $Q(Q, \psi)$ is medial iff $(Q, \cdot)$ is an abelian group
- $Q(Q, \psi)$ is distributive iff $(Q, \cdot)$ is a commutative Moufang loop and $\psi$ is 1-nuclear

**Lemma (Kepka-Němec)**

Let $(Q_1, \cdot), (Q_2, \cdot)$ be commutative Moufang loops, $\psi_1, \psi_2$ their 1-nuclear automorphisms. TFAE:
- $Q(Q_1, \psi_1) \simeq Q(Q_2, \psi_2)$
- there is a loop isomorphism $\rho : Q_1 \simeq Q_2$ such that $\psi_2 = \rho \psi_1 \rho^{-1}$
Enumeration

Theorem (Hou 2012)

- \( MI(p) = p - 2 \)
- \( MI(p^2) = 2p^2 - 3p - 1 \)
- \( MI(p^3) = 3p^3 - 6p^2 + p \)
- \( MI(p^4) = 5p^4 - 9p^3 + p^2 - 2p + 1 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>3^2</th>
<th>3^3</th>
<th>3^4</th>
<th>3^5</th>
<th>3^6</th>
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<td>CML*( (n) )</td>
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<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>≥ 8</td>
</tr>
<tr>
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<td>0</td>
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</tr>
<tr>
<td>D*( (n) )</td>
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<td>0</td>
<td>0</td>
<td>6</td>
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<tr>
<td>DS*( (n) )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
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<td>30</td>
<td>166</td>
<td></td>
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Here \( X^*(n) = X(n) - MI(n) \).