Every quasigroup is a factor of a subdirectly irreducible quasigroup

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Loops'07

Result

Theorem (Ralph McKenzie, DS)
Every quasigroup **Q** *is isomorphic to the factor of a subdirectly irreducible quasigroup* **R** *over its monolithic congruence* μ.

$$\forall \mathbf{Q}\ \exists \mathbf{R}\ \mathrm{SI}\ \textit{such that}\ \mathbf{Q} \simeq \mathbf{R}/\mu$$

- **Q** finite $\Rightarrow$ **R** finite
- **Q** (Bol) loop $\Rightarrow$ **R** (Bol) loop
- **Q** group $\Rightarrow$ **R** group

Construction

- $\mathbf{Q} = (Q, +)$ is the quasigroup
- $\mathbf{S} = (S, \cdot)$ is a simple non-Abelian group
- $\mathbf{R}$:= the *wreath product* of $\mathbf{S}$ and $\mathbf{Q}$

$$\mathbf{R} = \mathbf{S}^{(Q)} \rtimes \mathbf{Q}$$

$$(f, c) * (g, d) = (f \cdot (g \circ L_c), c + d)$$

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Fact. $\mathbf{R}$ is SI and μ is the kernel of the projection onto $\mathbf{Q}$.

$$\mathbf{Q} \simeq (\mathbf{S}^{(Q)} \rtimes \mathbf{Q})/ker\pi_2$$

Motivation: which algebras appear as SI/monolith ?

I. Assume at least one at least binary operation.

Observation

[Kepka 1981] $\mathbf{A} \simeq \mathbf{B}/\mu \Rightarrow$ *intersection of ideals is nonempty (IIN).*

Results

- [S.; Ježek–Kepka 2000] $\forall \mathbf{A}$ *with (IIN)* $\exists \mathbf{B}$ *SI s.t.* $\mathbf{A} \simeq \mathbf{B}/\mu$.
- [Bulman-Fleming–Hotzel–Wang 2004]
  $\forall \mathbf{A}$ *semigroup with (IIN)* $\exists \mathbf{B}$ *SI semigroup s.t.* $\mathbf{A} \simeq \mathbf{B}/\mu$.
- [Freese 2004] $\forall \mathbf{A}$ *lattice* $\exists \mathbf{B}$ *SI lattice s.t.* $\mathbf{A} \simeq \mathbf{B}/\mu$.
- [McKenzie–S. 2004]
  $\forall \mathbf{A}$ *quasigroup etc.* $\exists \mathbf{B}$ *SI quasigroup etc. s.t.* $\mathbf{A} \simeq \mathbf{B}/\mu$.

Motivation: which algebras appear as SI/monolith ?

II. All operations unary.

Observation

$\mathbf{A} \simeq \mathbf{B}/\mu \Rightarrow$ *intersection of* $\geq$ *2-element ideals is nonempty (2IIN).*

Results

- *Monounary algebras: exercise.*
- [Ježek–Marković–S. 2004]
  $\forall \mathbf{A}$ *finite multiunary with (2IIN)* $\exists \mathbf{B}$ *SI s.t.* $\mathbf{A} \simeq \mathbf{B}/\rho$.
- *Infinite case ???* $\quad$ *SI*$/\mu$ *???*

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