Is there an interesting concept of "binary abelianess"?

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A collapse in Mal'tsev hierarchy

Fact (Barto, Kozik, S.)

For finite idempotent algebras,

1. *abelian $\Rightarrow$ hereditarily absorption-free (HAF)*
2. *HAF & Taylor term $\Rightarrow$ Mal'tsev term*

$\mathbf{B} \le \mathbf{A}$ is absorbing = there is a term t such that, in every coordinate,

$$t(B, ..., B, A, B, ..., B) \subseteq B$$

HAF = no subalgebra has a proper absorbing subalgebra

abelian = for every term t, elements $x, y, \bar{u}, \bar{v}$,

$$t(x, \bar{u}) = t(x, \bar{v}) \Rightarrow t(y, \bar{u}) = t(y, \bar{v})$$

Corollary (Hobby, McKenzie, Theorem 9.6)

For a locally finite variety $\mathcal{V}$ with a Taylor term, there is a term that is Mal'tsev on blocks of solvable congruences of finite algebras in $\mathcal{V}$.

A collapse in Mal'tsev hierarchy, bin-case

Bin-Fact

For finite idempotent algebras,

1. *bin-abelian $\Rightarrow$ hereditarily bin-absorption-free (HbinAF)*
2. *HbinAF & Taylor term $\Rightarrow$ cube term*

HbinAF = no subalgera has a proper absorbing subalgebra with a binary t

bin-abelian = for every binary term t, elements a, b,

$$t(a, t(a, b)) = t(a, t(b, b)) \Rightarrow t(b, t(a, b)) = t(b, t(b, b))$$

Bin-Corollary

For a locally finite variety $\mathcal{V}$ with a Taylor term, there is a term that is cube on blocks of bin-solvable congruences of finite algebras in $\mathcal{V}$.

Bin-abelianess?

Term condition (TC):

$$t(x,\bar{u}) = t(x,\bar{v}) \Rightarrow t(y,\bar{u}) = t(y,\bar{v})$$

The condition we need to prove the Bin-Fact:

$$t(a,t(a,b)) = t(a,t(b,b)) \Rightarrow t(b,t(a,b)) = t(b,t(b,b))$$

A special instance of a more general / interesting condition?

- TC for binary t and arbitrary x, y, u, v
- TC for binary t and $x, y, u, v \in \langle a, b\rangle$
- TC for binary polynomial t and $\{x, y, u, v\} = \{a, b\}$
- dtto for an arbitrary t
- ... ?

Abelianess is important

Why abelianess:

- (polynomial subreducts of) modules are abelian
- abelian & has ..... term $\Rightarrow$ polynomial subreduct of a module
- abelian & Mal'tsev term $\Leftrightarrow$ polynomially equivalent to a module

Related notions of commutator, solvability, nilpotence.

For (finite) Mal'tsev algebras:

- nilpotent $\Leftrightarrow$ iterated central extensions
- solvable $\Leftrightarrow$ no polynomial subreduct that is a Boolean algebra
- some computational problems parametrized by an algebra:
  solvable $\Rightarrow$ difficult   nilpotent $\Rightarrow$ easy
- ...

In the "binary" case?

Is bin-abelianess important?

Idea: Gumm-Smith theorem with equations in two variables

di-module over a ring **R**:
$(M, +, -, 0, r\cdot)$ such that every 2-generated subalgebra is a module

Note: all axioms except associativity in 2 variables

[Kinyon, Vojtěchovský] commutative diassociative loops are a lot like abelian groups, e.g., decompose to p-components

Fact

An algebra is term equivalent to a di-module iff

1. *it has a Mal'tsev term and a term definable constant*
2. *the clone of term operations is generated by binary operations*
3. *it satisfies TC for every t and $x, y, \bar{u}, \bar{v}$ with $|\{x, y, \bar{u}, \bar{v}\}| \leq 2$*

Some examples

Commutative Moufang loops (CML):

- commutative, diassociative, inner mappings are automorphisms
- hence di-modules over $\mathbb{Z}$
- *Moufang di-modules* seem to be important

Representing pointed quasigroups:

- medial iff term equivalent to a pointed module over $\mathbb{Z}[x, y]$ [Toyoda-Bruck]
- trimedial iff term equivalent to a centrally pointed Moufang di-module over $\mathbb{Z}[x, y]$ [Kepka]

Perhaps bin-abelianess shall be stronger, to allow only Moufang di-modules?

Other ideas

Or, perhaps bin-abelianess shall be weaker, restricting term arity in TC?

- a representation with some very weak notion of a module?
- a representation with a structure that only requires cube term? (not Mal'tsev)

solvable $\Leftrightarrow$ no subreduct that is a Boolean algebra

bin-solvable $\Leftrightarrow$ no subreduct that is ???

Ballot

The idea to introduce some sort of bin-abelianess is

1. WORTHWHILE
2. WORTHLESS
3. Good morning, what is abelianess?