Subdirectly irreducible algebras in a class of strongly solvable modes

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June 2011, Kraków

Park's conjecture

residual bound of $\mathcal{V}$ = the smallest cardinal such that all subdirectly irreducibles in $\mathcal{V}$ have size $< \kappa$

Problem (Park's conjecture)

Consider a variety that has

- *finite signature*
- *finite residual bound*

Does it have a finite base for its equations?

YES, if $\mathcal{V}$ is congruence modular, or congruence SD($\wedge$)

Residual bounds for finitely generated varieties

Theorem (R. McKenzie)

Finitely generated varieties have residual bound finite, or $\aleph_0$, or $\aleph_1$, or $(2^{\aleph_0})^+$, or there is no bound at all.

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??? What if we require finite signature ???

… finite, $\aleph_1$, $(2^{\aleph_0})^+$, no bound — YES

Problem (RS problem)

Is there one with $\aleph_0$?

… NO, if $\mathcal{V}$ is congruence modular, or congruence SD($\wedge$)

Modes

modes = idempotent algebras with commuting term operations

Theorem (K. Kearnes)

Let $\mathcal{V}$ be a finitely generated variety of modes. Then

$$\mathcal{V} = (\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5,$$

where

- $\mathcal{V}_1$ *is strongly solvable variety*
- $\mathcal{V}_2$ *is equivalent to a variety of modules over a commutative ring*
- $\mathcal{V}_5$ *has a semilattice term*

... RS problem, Park's conjecture, abelian iff quasi-affine... for modes
... what is $\mathcal{V}_1$?

Differential modes

differential mode = a ternary mode $\mathbf{A}$ with a congruence α such that

- all blocks of α are left projections
- the factoralgebra $\mathbf{A}/\alpha$ is left projection

Examples:

- $(M, \_)$ with $(xyz) = (1-a-b)x + ay + bz$ on a module over a commutative ring, where $a^2 = b^2 = ab = 0$
- $(\{0,1,2\}, \_)$ with $(xyz) = \begin{cases} 2-x & \text{if } y = z = 1, \\ x & \text{otherwise.} \end{cases}$

Fact

- *Differential modes form a variety.*
- *Every differential mode has a strongly solvable chain* $0 \le \lambda \le 1$.

... $a \; \lambda \; b$ iff there are right translations t, s such that $t(a) = s(b)$

Subdirectly irreducibles I

Let $\mathbf{A}$ be an *SI differential mode*. Then

- λ has exactly one non-trivial block B,
- hence $\mathbf{A} = \mathbf{B} \propto \mathbf{C} = (B \cup C, \_)$ with

$$(c\_\_) = c,\ (bb_1b_2) = b,\ (bb_1c) = g_c(b),\ (bcb_1) = h_c(b),\ (bcd) = f_{cd}(b)$$

- and (B, f_{cd}, g_c, h_c) is an *SI commutative unary algebra*

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Theorem (Ésik and Imreh)

Every SI commutative unary algebra is of one of the three types:

(I) *cocyclic* $=$ *equivalent to a* $\mathbb{Z}_{p^k}$*-set* $(k = 1, 2, \ldots, \infty)$

(II) *cocyclic+1* $=$ *cocyclic* $\cup$ *singleton*

(III) *nilpotent* $=$ *see [PICTURE]*

Subdirectly irreducibles II

Theorem

$\mathbf{A}$ *is a proper SI differential mode if and only if* $\mathbf{A} = \mathbf{B} \propto \mathbf{C}$ *and*

- *for the unary algebra* $(B, \_)$, *one of the two options takes place:*
  - (I) *it is cocyclic*
  - (III) *it is nilpotent, and for every* $c \in C$ *at least one of the following takes place:* $f_{cc} \neq id$, $g_c \neq id$, $h_c \neq id$, $f_{cd} \neq g_d$ *for some* d, $f_{dc} \neq h_d$ *for some* d
- *for every* $c \neq d$, *at least one of the following takes place:* $g_c \neq g_d$, $h_c \neq h_d$, $|\{f_{cc}, f_{dd}, f_{cd}, f_{dc}\}| > 1$, $f_{ce} \neq f_{de}$ *for some* e, $f_{ec} \neq f_{ed}$ *for some* e

Residual bounds for differential modes

Szendrei mode = admits a linear representation over semimodules

$\ldots\ (xyz) = ((xyx)xz) \qquad \Longrightarrow$ in $\mathbf{B} \propto \mathbf{C}$, $f_{cd} = g_d h_c$

Theorem

1. *Every non-Szendrei variety of differential modes is residually large.*
2. *A Szendrei variety has a finite residual bound iff [... $l \leq 1$, $p < \infty$ ...].*
3. *A locally finite Szendrei variety failing [...] is residually large.*

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Proof:

1. Every non-Szendrei variety contains a finite SI such that
   - it is non-Szendrei cocyclic and $|C| \leq 2$; or
   - it is non-Szendrei nilpotent of length 1 and $|C| \leq 2$; or
   - it is nilpotent of length 2 with trivial blocks, linear order and $|C| \leq 2$.

   Find a construction in each case.
2. in the Szendrei case, $|C| < |B|^{2|B|}$
3. apply item three of (1)

Park's conjecture for differential modes

Corollary

Every variety of differential modes with a finite residual bound is finitely based.

Proof:

... it is a Szendrei variety

... Szendrei varieties correspond to congruences of $(\mathbb{N}, +) \times (\mathbb{N}, +)$

... all such congruences are finitely generated by Rédei's theorem