Selfdistributive One-sided Quasigroups

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Two problems by Tomáš Kepka

1. Homomorphic images of subdirectly irreducible algebras
   - Problem: classify $A \in \mathcal{V}$ such that $A \simeq B/\mu$ for a $B \in \mathcal{V}_{SI}$
   - (Kepka 1981) an obvious necessary condition
   - the condition is sufficient for
     - (S.; Ježek–Kepka 2000) $\mathcal{V} =$ algebras of rich signature
     - (Bulman-Fleming–Hotzel–Wang 2004) $\mathcal{V} =$ semigroups
     - (McKenzie–S. 2004) $\mathcal{V} =$ quasigroups, loops, groups
     - (Ježek–Marković–S. 2004) $\mathcal{V} =$ finite unary algebras (μ not necessarilly monolithic)

2. Left distributive left quasigroups
   - $=$ groupoids where all left translations are automorphisms
     - $x * (y * z) = (x * y) * (x * z)$
     - $a * x = b$ has a unique solution for all a, b
   - Problem: understand their structure

Left distributive left quasigroups

= groupoids where all left translations are automorphisms

$x * (y * z) = (x * y) * (x * z)$

$a * x = b$ has a unique solution for all a, b

Idempotent LDLQs:

- *Group conjugation:* on a group (or a conjugacy class), put
$$x * y = xyx^{-1}.$$
- *Linear groupoids:* on a module over a commutative ring, put
$$x * y = (1 - k)x + ky.$$
- *Knot quandles:* a classifying invariant of knot equivalence

Non-idempotent LDLQs:

- no natural examples
- combinatorial objects over idempotent ones

Non-idempotent LDLQs: the ip congruence

Circle of length $n = 1, 2, \ldots, \infty$: $\quad x * y = y + 1 \mod n$

Fact

Let ip be the smallest congruence such that G/ip is idempotent. Then

- $ip = Equiv(\{(a, aa) : a \in G\})$
- $ip = \{(a, b) \in G \times G : a^m = b^n \text{ for some } m, n\}$
- *every block of ip is a circle*

Hence, in a sense, non-idempotent LDLQs result form idempotent LDLQs by replacing elements for circles.

Consequences:

- description of free groupoids, normal forms in some varieties
- description of the subvariety lattice
- description of subdirectly irreducibles

All up to the idempotent case.

Non-idempotent LDLQs: free groupoids, varieties

Exponent n means the variety satisfies $x^{n+1} = x$.
$\mathbf{F}_{\mathcal{V}}(X)$ denotes the free algebra over X in $\mathcal{V}$.
$\mathcal{I}$ denotes the variety of idempotent algebras.

Theorem

Let $\mathcal{V}$ be a variety of LD left quasigroups of exponent n.

- $\mathbf{F}_{\mathcal{V}}(X) \simeq C_n \times \mathbf{F}_{\mathcal{V}\cap\mathcal{I}}(X)$, *where C_n is the circle of length n*
- *The lattice of subvarieties of $\mathcal{V}$ is isomorphic to the lattice*

$$(L \times \{1\}) \cup (K \times (N \setminus \{1\})) \;\subseteq\; L \times N,$$

where L is the lattice of subvarieties of $\mathcal{V} \cap \mathcal{I}$, K its sublattice of varieties containing right zero bands and N the lattice of positive integer divisors of n.

Non-idempotent LDLQs: subdirectly irreducibles

Simple: either idempotent, or isomorphic to a circle of prime length

SI:

Theorem

Let G be a non-idempotent subdirectly irreducible LDLQ.

- *all ip-blocks are of size 1 or p^k*
- *term-definable left division $\Rightarrow$ the monolith is the smallest congruence such that G/μ has exponent p^{k-1}*
- *G embeds into $K \cup \mathrm{Aut}(K)$, where $K = \{a : aa \neq a\}$ and the operation is defined by*

$$\begin{array}{c|cc} * & v & \psi \\ \hline u & uv & L_u\psi(L_u)^{-1} \\ \varphi & \varphi(v) & \varphi\psi\varphi^{-1} \end{array}$$

Idempotent LDLQs: equational theory

Idempotent LDLQs are closely related to groups, via conjugation.

Fact (Drápal, Kepka, Musílek)

The following varieties of groupoids coincide:

- *the variety generated by idempotent LDLQs*
- *the variety generated by conjugation groupoids of groups*

Equational basis: open problem!!!
The simplest equation that doesn't follow from LD,I:

$$((xy)y)(xz) = (xy)((yx)z)$$

Fact (Joyce)

In signature $\{\cdot, \backslash\}$, *the equational basis is just*

$$xx = x, \quad x(yz) = (xy)(xz), \quad x\backslash(xy) = x(x\backslash y) = y$$

Idempotent LDLQs: left multiplication group

Left translation: $L_a : x \mapsto ax$
Left multiplication group: $\mathrm{LMlt}(G) = \langle L_a : a \in G \rangle$

Observation

$a \mapsto L_a$ is a homomorphism from G into the conjugation groupoid of $\mathrm{LMlt}(G)$. Proof: LD says $L_{a*b} L_a = L_a L_b$.

Consequence: many properties translate into/from groups.
E.g., simple quandles relate to simple groups:

Theorem (Joyce)

Every finite simple idempotent LDLQ is isomorphic to $Q\langle G, C, m \rangle$, where G is a finite simple group, C a conjugacy class in $\mathrm{Aut}(G)$ and $m \geq 1$ (all uniquely determined). Here $Q\langle G, C, m \rangle$ denotes the conjugation groupoid of a conjugacy class in [...a group constructed using G, C, m...]

The knot quandle

Joyce/Matveev (1980's):

$$\begin{array}{c}\text{an oriented tame knot}\\ \downarrow \\ \text{a regular projection to 2D} \\ \downarrow \\ \text{the knot quandle} = \langle \text{ arcs: relations } \rangle\end{array}$$

Relations:

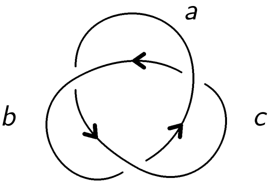
- $xy = z$ for every underpass, where "x coming under y becomes z" and y is going over x in the left-right direction
- $x\backslash y = z$ dtto with right-left direction

Theorem (Joyce/Matveev)

Two tame knots are equivalent, iff their knot quandles are isomorphic.

The knot quandle: examples

$$Q = \langle \text{ arcs : relations "}x\text{ coming under }y\text{ becomes }z\text{" } \rangle$$



The *trefoil* quandle is

$$\langle a, b, c \;:\; ab = c,\; bc = a,\; ca = b\rangle \simeq \mathrm{Core}(\mathbb{Z}_3)$$

Here $\mathrm{Core}(G) = (G, 2x - y)$.

The knot quandle: examples

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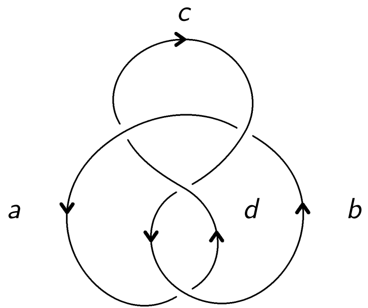

The the *figure-eight* quandle is

$$\langle a, b, c, d \ : \ ab = d,\ b\backslash c = a,\ cd = b,\ d\backslash a = c \rangle \simeq \mathrm{Core}(\mathbb{Z}_5)$$

The knot quandle: inspiration for further research

Several papers by knot theorists on idempotent LDLQs appeared recently. They seem to be interested in

- computing in LDLQs
- classifying small LDLQs

Mutual cooperation is desirable.