Differential modes

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Modes

n-ary mode = an idempotent entropic algebra $\mathbf{A} = (A, f)$

Idempotency:

$$f(x, x, \dots, x) = x$$

Entropy:

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) = \\ f(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1n}, \dots, x_{nn}))$$

- f is a homomorphism $\mathbf{A}^n \to \mathbf{A}$
- the clone of term operations is commutative

Differential modes: semantic definition

LZ algebra $= (A, f)$ with $f(x_1, \ldots, x_n) = x_1$

(n-ary left) differential mode $=$ a mode $\mathbf{A}$ possessing a congruence α such that

- all blocks of α are LZ
- the factoralgebra $\mathbf{A}/\alpha$ is LZ

It means, the variety $\mathcal{D}_n$ of differential modes is the Mal'tsev product

$$\mathcal{LZ}_n \circ \mathcal{LZ}_n$$

relative to modes.

The structure of finite modes

Theorem (Kearnes)

Let $\mathcal{V}$ be a finitely generated variety of modes. Then

$$\mathcal{V} = (\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5,$$

where

- $\mathcal{V}_1$ *is strongly solvable variety*
- $\mathcal{V}_2$ *is equivalent to a variety of modules over a commutative ring*
- $\mathcal{V}_5$ *is a variety of semilattice modes*

Conjecture. $\mathcal{V}_1$ is a Mal'tsev product of projection algebras.

Examples

- (Romanowska, Smith) on a differential group, put

$$x * y = x - dx + dy$$

- *Linear modes:* on a module over a commutative ring $\mathbf{R}$, pick $k \in R$ such that $k^2 = 0$, and put

$$x * y = (1-k)x + ky$$

(e.g., $\mathbf{R} = \mathbb{Z}_4$, $k = 2$, or $\mathbf{R} = \mathbb{Z}[k]/k^2$)

- the ternary algebra $(\{0, 1, 2\}, f)$ with

$$f(x, y, z) = \begin{cases} 2 - x & \text{if } y = z = 1, \\ x & \text{otherwise.} \end{cases}$$

WHY differential modes?

Embedding modes into semimodules over commutative semirings

- for a long time, it wasn't known whether all modes are embeddable
- modules obviously are, semilattice modes are (Kearnes)
- all differential groupoids are
- the last (ternary) example is not
- in fact, many differential modes are not

Embedding algebras into modules: Do abelian algebras embed?

- generally not
- yes for nice varieties (CM, ...)
- for modes? differential modes?

Park's conjecture: Is every residually finite variety finitely based?

- true for nice varieties (CD, CM, ...)
- for ugly varieties? (modes? differential modes?)

Differential modes: syntactic definition

Idempotency:

$$f(x, x, \dots, x) = x$$

Entropy:

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) = \\ f(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1n}, \dots, x_{nn}))$$

Left reductive law:

$$f(x, f(y_{21}, y_{22}, \dots, y_{2n}), \dots, f(y_{n1}, y_{n2}, \dots, y_{nn})) = f(x, y_{21}, \dots, y_{n1})$$

Left normal law:

$$f(f(x, y_2, \dots, y_n), z_2, \dots, z_n) = f(f(x, z_2, \dots, z_n), y_2, \dots, y_n)$$

Ternary case.

Useful shortcuts:

$$(xyz) = f(x, y, z)$$

$$(xyz)^k = (((xyz)yz)\dots yz)$$

Let $(x_1, y_1), \dots, (x_m, y_m)$ be a list of elements of $X \times X$.

Fact

Every term over X is equivalent to a unique term of the form

$$((\dots((xx_1y_1)^{k_1}x_2y_2)^{k_2}\dots)x_my_m)^{k_m},$$

where $k_i = 0$ for the pair (x, x).

Embeddings into semimodules over commutative semirings

Reduct of a semimodule = *semiaffine representation* = operations are semimodule terms, i.e.,

$$f(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

Subreduct = subalgebra of a reduct

Theorem (Stronkowski, S.)

A mode is a subreduct of a semimodule over a commutative semiring if and only if it satisfies the Szendrei identities.

Szendrei identities:

$$f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})) = \\ f(f(x_{\pi(11)}, \ldots, x_{\pi(1n)}), \ldots, f(x_{\pi(n1)}, \ldots, x_{\pi(nn)}))$$

where $\pi : ij \leftrightarrow ji$ for a single fixed ij.

Free Szendrei differential modes

Ternary case.

$$((xx_1y_1)x_2y_2) = ((xx_2y_1)x_1y_2)$$

$$((xx_1y_1)x_2y_2) = ((xx_1y_2)x_2y_1)$$

Let $x_1, \dots, x_n$ be the list of the elements of X.

Fact

Every term over X is equivalent to a unique term of the form

$$((\dots((xxx_1)^{k_1}x_1x)^{l_1}xx_2)^{k_2}x_2x)^{l_2}\dots)xx_n)^{k_n}x_nx)^{l_n},$$

where $k_il_i = 0$ for all i, and $k_j = l_j = 0$ for $x_j = x$.

Embeddings into modules over commutative rings

Reduct of a module = *affine representation* = operations are module terms, i.e.,

$$f(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

Subreduct = subalgebra of a reduct = *quasi-affine algebras*

Observation

Quasi-affine algebras are abelian, i.e., for every term t

$$t(x, u_1, \ldots, u_k) = t(x, v_1, \ldots, v_k) \Rightarrow t(y, u_1, \ldots, u_k) = t(y, v_1, \ldots, v_k)$$

Proved: Abelian differential groupoids are subreducts of modules.

Conjectured: Abelian differential modes are subreducts of modules.

Question: Are abelian modes subreducts of modules?
(True for nice varieties, false generally, unknown for modes.)

The lattice of subvarieties

Easy fact: Every subvariety of differential modes

- either, has a relative base consisting of 1 additional equation,
- or, has no finite base.

Not so easy fact: There exist subvarieties with no finite basis, e.g.,

$$(((xyz)yz)yz) = ((xyz)yz)$$

$$(((((xxx_1)x_1x_2)x_2y) = (((((xxx_1)x_1x_2)yx_2)$$

$$(((((xxx_1)x_1x_2)x_2x_3)x_3y) = (((((xxx_1)x_1x_2)x_2x_3)yx_3)$$

$$\dots$$

$$(((((xxx_1)x_1x_2)\dots)x_{n-1}x_n)x_ny) = (((((xxx_1)x_1x_2)\dots)x_{n-1}x_n)yx_n)$$

$$\dots$$

The lattice of subvarieties, the Szendrei case

In contrast, subvarieties of Szendrei differential modes are tame:

- every one can be relatively based in two variables,
- the subvarieties correspond 1-1 to congruences of $(\mathbb{N}, +)^{n-1}$, or

$$L \simeq \mathbf{Con}(\mathbb{N}, +)^{n-1}$$

- consequently, by Rédei's theorem, all are finitely based.

Open problems

1. Finitely generated strongly solvable varieties of modes.
2. Abelian implies quasi-affine for (differential) modes.
3. Park's conjecture for (differential) modes.

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