Homomorphic images of subdirectly irreducible algebras

David Stanovský

Charles University in Prague, Czech Republic
An algebra is called *subdirectly irreducible* (SI), if the intersection of non-trivial congruences, called the *monolith* \( \mu \), is non-trivial.

Every algebra can be embedded into the product of SI algebras (from the variety it generates).

*Which algebras are homomorphic images of SI algebras?*

*Example:* \( \mathbb{Z}(+) \) is not. Every groupoid with 0 is (by Kepka, 1981).
Necessary condition.

An *ideal* in $A$ is a subset $\emptyset \neq I \subseteq A$ such that $f(a_1,\ldots,a_n) \in I$, whenever at least one $a_i \in I$, for every operation $f$ on $A$.

It is easy to see that a non-unary SI algebra has a non-empty intersection of its ideals, and an inverse image of an ideal is an ideal.

So, *any homomorphic image of a non-unary SI algebra has a non-empty intersection of its ideals.*

*Example:* $\mathbb{Z}(\oplus)$ has empty intersection of its ideals.
Rich signatures.

Theorem 1. (S. 2001; Kepka, Ježek 2002) Assume an algebra $A$ has at least one at least binary operation. TFAE:

1. $A$ is a homomorphic image of an SI algebra $B$;

2. $A$ is isomorphic to $B/\mu$ for an SI algebra $B$;

3. $A$ has a non-empty intersection of its ideals.

Moreover, if $A$ is finite, then $B$ can be taken finite.
Monounary algebras.

Note that in unary algebras ideal = subalgebra.

Theorem 2. Let $A$ be a monounary algebra, $|A| \geq 2$. TFAE:

1. $A$ is a homomorphic image of an SI algebra;

2. $A$ is isomorphic to $B/\mu$ for an SI algebra $B$;

3. $A$ is SI;

4. $A$ is either a path, or a circle of prime power length, possibly with an extra point, or $|A| = 2$. 
Unary algebras.

**Theorem 3.** (Ježek, Marković, S.) Let $A$ be a finite unary algebra with at least two operations. TFAE:

1. $A$ is a homomorphic image of an SI algebra;

2. $A$ has a non-empty intersection of its at least 2-element subalgebras;

3. $A$ has a smallest subalgebra or $A$ has two disjoint subalgebras $U, \{a\}$ and $U$ is the smallest subalgebra of $A - \{a\}$.

Moreover the SI algebra can be taken finite.
Problem 1. What are homomorphic images of infinite SI unary algebras?

Problem 2. Is there a nice characterization of (finite) SI unary algebras, which are isomorphic to $B/\mu$ for an SI $B$?
The same problem in a particular variety

Assume $A$ is an algebra with a non-empty intersection of ideals from a variety $\mathcal{V}$.

Is it a homomorphic image (over the monolith) of an $SI$ algebra from the variety $\mathcal{V}$?

Certainly not always — e.g. consider the variety of distributive lattices.

The construction for algebras with a rich signature keeps idempotency. In fact, if $A$ satisfies $t(x) \approx x$, then the $SI$ $B$ satisfies $t(x) \approx x$. And if $A$ has no proper ideals, the construction keeps any identity in one variable. (And probably nothing else.)
For some well-known particular varieties? Like groups, lattices, semigroups, rings?
Semigroups.

**Theorem 4.** (Bulman-Fleming, Hotzel, Wang 2004) Let $A$ be a semigroup. TFAE:

1. $A$ is a homomorphic image of an SI semigroup;

2. $A$ is isomorphic to $B/\mu$ for an SI semigroup $B$;

3. $A$ has a non-empty intersection of its ideals.

However, if $A$ is finite, $B$ is *not always finite!!!* (E.g., for right zero semigroups there is no such finite $B$.)
Lattices and Quasigroups.

Note that every quasigroup and every lattice has no proper ideal.

**Theorem 5.** (Freese) Every lattice is isomorphic to the factor of an SI lattice over its monolith. For a finite one, the SI can be chosen finite.

**Theorem 6.** (McKenzie, S.) Every group, loop, quasigroup is isomorphic to the factor of an SI group, loop, quasigroup over its monolith. For a finite one, the SI can be chosen finite.

*Proof.* Use the wreath product with a simple non-abelian group.
The commutative case.

For *abelian groups*, the task is easy: the only SIs are $\mathbb{Z}_{p^k}$, $k = 1, \ldots, \infty$.

*Problem 3.* Is every *commutative* loop, quasigroup a homomorphic image of an SI commutative loop, quasigroup?

*Problem 4.* What about *commutative groupoids*?
References.


