Homomorphic images of subdirectly irreducible algebras

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An algebra is called *subdirectly irreducible* (SI), if the intersection of non-trivial congruences, called the *monolith* μ, is non-trivial.

Every algebra can be embedded into the product of SI algebras (from the variety it generates).

Which algebras are homomorphic images of SI algebras?

Example: $\mathbb{Z}(+)$ is not. Every groupoid with 0 is (by Kepka, 1981).

Necessary condition.

An *ideal* in $\mathbf{A}$ is a subset $\emptyset \neq I \subseteq A$ such that $f(a_1, \ldots, a_n) \in I$, whenever at least one $a_i \in I$, for every operation f on $\mathbf{A}$.

It is easy to see that a non-unary SI algebra has a non-empty intersection of its ideals, and an inverse image of an ideal is an ideal.

So, *any homomorphic image of a non-unary SI algebra has a non-empty intersection of its ideals.*

Example: $\mathbb{Z}(+)$ has empty intersection of its ideals.

Rich signatures.

Theorem 1. (S. 2001; Kepka, Ježek 2002) *Assume an algebra* $\mathbf{A}$ *has at least one at least binary operation. TFAE:*

1. $\mathbf{A}$ *is a homomorphic image of an SI algebra* $\mathbf{B}$*;*

2. $\mathbf{A}$ *is isomorphic to* $\mathbf{B}/\mu$ *for an SI algebra* $\mathbf{B}$*;*

3. $\mathbf{A}$ *has a non-empty intersection of its ideals.*

Moreover, if $\mathbf{A}$ *is finite, then* $\mathbf{B}$ *can be taken finite.*

Monounary algebras.

Note that in unary algebras ideal = subalgebra.

Theorem 2. *Let* $\mathbf{A}$ *be a monounary algebra*, $|A| \geq 2$. *TFAE:*

1. $\mathbf{A}$ *is a homomorphic image of an SI algebra;*

2. $\mathbf{A}$ *is isomorphic to* $\mathbf{B}/\mu$ *for an SI algebra* $\mathbf{B}$*;*

3. $\mathbf{A}$ *is SI;*

4. $\mathbf{A}$ *is either a path, or a circle of prime power length, possibly with an extra point, or* $|A| = 2$.

Unary algebras.

Theorem 3. (Ježek, Marković, S.) *Let $\mathbf{A}$ be a finite unary algebra with at least two operations. TFAE:*

1. *$\mathbf{A}$ is a homomorphic image of an SI algebra;*

2. *$\mathbf{A}$ has a non-empty intersection of its at least 2-element subalgebras;*

3. *$\mathbf{A}$ has a smallest subalgebra or $\mathbf{A}$ has two disjoint subalgebras $U, \{a\}$ and U is the smallest subalgebra of $A - \{a\}$.*

Moreover the SI algebra can be taken finite.

Problem 1. What are homomorphic images of *infinite* SI unary algebras?

Problem 2. Is there a nice characterization of (finite) SI unary algebras, which are *isomorphic to* $\mathbf{B}/\mu$ *for an SI* $\mathbf{B}$?

The same problem in a particular variety

Assume $\mathbf{A}$ is an algebra with a non-empty intersection of ideals *from a variety* $\mathcal{V}$.

Is it a homomorphic image (over the monolith) of an *SI* algebra *from the variety* $\mathcal{V}$?

Certainly not always — e.g. consider the variety of distributive lattices.

The construction for algebras with a rich signature keeps idempotency. In fact, *if* $\mathbf{A}$ *satisfies* $t(x) \approx x$, *then the SI* $\mathbf{B}$ *satisfies* $t(x) \approx x$. And if $\mathbf{A}$ has no proper ideals, the construction keeps any identity in one variable. (And probably nothing else.)

For some well-known particular varieties? Like *groups, lattices, semigroups, rings*?

Semigroups.

Theorem 4. (Bulman-Fleming, Hotzel, Wang 2004) *Let $\mathbf{A}$ be a semigroup. TFAE:*

1. *$\mathbf{A}$ is a homomorphic image of an SI semigroup;*

2. *$\mathbf{A}$ is isomorphic to $\mathbf{B}/\mu$ for an SI semigroup $\mathbf{B}$;*

3. *$\mathbf{A}$ has a non-empty intersection of its ideals.*

However, if $\mathbf{A}$ is finite, $\mathbf{B}$ is *not always finite!!!*
(E.g., for right zero semigroups there is no such finite $\mathbf{B}$.)

Lattices and Quasigroups.

Note that every *quasigroup* and every *lattice* has no proper ideal.

Theorem 5. (Freese) *Every lattice is isomorphic to the factor of an SI lattice over its monolith.*
For a finite one, the SI can be chosen finite.

Theorem 6. (McKenzie, S.) *Every group, loop, quasigroup is isomorphic to the factor of an SI group, loop, quasigroup over its monolith.*
For a finite one, the SI can be chosen finite.

Proof. Use the wreath product with a simple non-abelian group.

The commutative case.

For *abelian groups*, the task is easy: the only SIs are $\mathbb{Z}_{p^k}$, $k = 1, \ldots, \infty$.

Problem 3. Is every *commutative* loop, quasigroup a homomorphic image of an SI commutative loop, quasigroup?

Problem 4. What about *commutative groupoids*?

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