DIFFERENTIAL MODES

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Abstract. Modes are idempotent and entropic algebras. Although it had been established many years ago that groupoid modes embed as subreducts of semimodules over commutative semirings, the general embeddability question remained open until M. Stronkowski and D. Stanovský's recent constructions of isolated examples of modes without such an embedding. The current paper now presents a broad class of modes that are not embeddable into semimodules, including structural investigations and an analysis of the lattice of varieties.

It is well known that each entropic groupoid ("medial" in the terminology of Ježek-Kepka) with surjective operation embeds as a subreduct into a semimodule over a commutative semiring [1]. In particular, each idempotent and entropic groupoid, i.e. each groupoid mode (as defined e.g. in [12]) embeds into such a semimodule. (See [1] and [6]). Surprisingly, this is no longer true for modes with operations of larger arity. As shown by M. Stronkowski [17] and [18], a mode embeds as a subreduct into a semimodule over a commutative semiring if and only if it satisfies the so-called Szendrei identities. A simpler proof was then given by D. Stanovský [16]. Stronkowski also proved that free modes do not satisfy the Szendrei identities, while Stanovský [16] provided a 3-element example of a mode with one ternary operation (Example 1.1). In this paper we analyze Stanovský's example, and show that it belongs to the variety of so-called ternary differential modes,

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which form a ternary counterpart of the variety of differential groupoid modes [10]. We investigate properties of this variety, and show that it contains a broad class of modes not satisfying the Szendrei identities, i.e. not embeddable into semimodules over commutative semirings. To simplify notation, we consider only algebras with one ternary operation, but all our results may easily be extended to algebras with one basic operation of any arity $n > 3$.

Note that the possibility of embedding given algebras as subreducts into other "richer" algebras provides an efficient method for investigating their structure. In particular, if these richer algebras are (semi)modules, such an embedding allows us to represent operations as linear combinations, providing so-called linear representations for the algebras being embedded. The method appeared to be quite successful in investigating the structure of modes. Apart from the above-mentioned result of Ježek and Kepka (and a number of partial results preceding it), let us mention the result of K. Kearnes [2] that semilattice modes embed into semimodules over commutative semirings, and results of A. Romanowska, J.D.H. Smith and A. Zamojska-Dzienio [11], [13], [14], [20] showing that certain sums of cancellative modes embed into certain special semimodules over commutative semirings. We now know that not all modes have the embeddability property. Thus it becomes critical to locate the borderline between three classes of modes:

- those embeddable into modules over commutative rings;
- those embeddable into semimodules over commutative semirings;
- those that do not embed into semimodules.

An essential role is also played by modes equivalent to affine spaces over commutative rings, and modes equivalent to affine semimodules over commutative semirings. Recall that affine spaces are characterized as Mal'cev modes, and form full idempotent reducts of the corresponding modules, while affine semimodules form full idempotent reducts of the corresponding semimodules. Differential modes are well suited to investigations of the embedding problems.

The paper provides some results from a larger project that investigates these problems, and analyzes embeddability and non-embeddability of differential modes.

First we introduce the algebras in question, and show how they are related to differential groupoids (Sections 1 and 2). The main part of the paper concerns three topics. In Section 3, we show that each ternary differential mode has a homomorphism onto a left-zero algebra with left-zero subalgebras as the congruence classes of the corresponding kernel. Then we show that the mode can be reconstructed from these classes and the quotient by means of a special construction called an $Lz \circ Lz$-sum. We frequently use this construction in the subsequent work. Section 4 provides a characterization of absolutely free differential modes (Theorem 4.2), and of free differential modes in three subvarieties playing a special role in our investigations:

- The variety of Szendrei modes (those embeddable into semimodules over commutative semirings) (Theorem 4.4);
- The variety of so-called hemisemiprojection modes;
- The variety of semiprojection modes.

We also show that the only Szendrei hemisemiprojection modes are left-zero algebras. This provides a new class of modes not embeddable into semimodules (Corollary 4.6). The final Section 5 is devoted to varieties of differential modes. In contrast with differential groupoids, the lattice of varieties of ternary differential modes is much more complex. Though proper nontrivial finitely based varieties can also be defined by one more additional identity (Theorem 5.2), the number of variables in such identities grows rapidly, and there are varieties not having a finite basis for their identities (Theorem 5.5).

A deeper analysis of embeddability and nonembeddability of differential modes will be provided in a subsequent paper that will contain more information about geometrical aspects of differential modes, and a more detailed analysis of Szendrei and hemisemiprojection modes.

For further information concerning the theory of modes, we refer the reader to the two monographs [9] and [12]; for universal algebra, one may also consult standard books on the subject. We frequently follow the notation and terminology used in the two monographs on modes. In particular, we often use algebraic notation for functions and operations, reserving special notation for binary and ternary operations. The concepts "term" and "word" are synonymous, as are the concepts "term operation" and "derived operation".

1. Introduction

In [19], Á. Szendrei introduced certain identities which are satisfied by reducts of any type of affine spaces over a commutative ring R with identity. For a given type τ, these are identities arising from each word (term) of type τ of the form

$$x_{11} \dots x_{1n} w \ \dots \ x_{1n} \dots x_{nn} w \, w,$$

where w is a derived operator with n variables defining a basic operation of the reduct in question, by interchanging x_{ij} and x_{ji} for fixed $1 \leq i, j \leq n$. Note that these Szendrei identities are satisfied by all subreducts (subalgebras of reducts) of semimodules over commutative semirings. Semirings with identity and semimodules over such semirings are defined similarly as rings and modules, however with commutative semigroups replacing abelian groups. In this paper we consider only commutative semirings, and semimodules over such semirings. The idempotent subreducts of semimodules are obviously modes, i.e. they are idempotent and entropic (each singleton is a subalgebra and each operation is a homomorphism.) For example, in the case of one ternary operation $f(x, y, z) =: (xyz)$ there are three Szendrei identities:

$$((x_{11}X_{12}x_{13})(X_{21}x_{22}x_{23})(x_{31}x_{32}x_{33})) = ((x_{11}X_{21}x_{13})(X_{12}x_{22}x_{23})(x_{31}x_{32}x_{33})), \tag{1.1}$$

$$((x_{11}x_{12}X_{13})(x_{21}x_{22}x_{23})(X_{31}x_{32}x_{33})) = ((x_{11}x_{12}X_{31})(x_{21}x_{22}x_{23})(X_{13}x_{32}x_{33})), \tag{1.2}$$

$$((x_{11}x_{12}x_{13})(x_{21}x_{22}X_{23})(x_{31}X_{32}x_{33})) = ((x_{11}x_{12}x_{13})(x_{21}x_{22}X_{32})(x_{31}X_{23}x_{33})). \tag{1.3}$$

Example 1.1. An example we are interested in is the 3-element algebra (D, f), where $D = \{0, 1, 2\}$, with one ternary operation $f : D^3 \to D; (a, b, c) \mapsto f(a, b, c) =: (abc)$. The operation f is defined by

$$(abc) := \begin{cases} 2 - a, & \text{if } b = c = 1 \\ a & \text{otherwise.} \end{cases}$$

It is easy to check that the algebra (D, f) is a mode, but does not satisfy the identity (1.2). Indeed, $((210)(000)(100)) = (201) = 2 \neq 0 = (000) = ((211)(000)(000))$. Hence it is not embeddable into a semimodule over a commutative semiring.

Note that the algebra (D, f) is an "almost left-zero" algebra. It differs from a left-zero algebra only in two places: $(011) = 2$ instead of 0 and $(211) = 0$ instead of 2.

Let us call an algebra (A, f) with an n-ary operation f an *i-zero* or *i-trivial* or just a *projection algebra* if the operation f is the i-th projection. Note that an n-dimensional diagonal algebra (A, f) is always a direct product of n trivial (projection) subalgebras, one 1-zero (or *left-zero*) algebra, one 2-zero algebra, and so on. (See [4].) (An n-zero algebra will also be called a *right-zero* algebra.) Such algebras are embeddable into modules over commutative rings (see e.g. [20]), whence they satisfy Szendrei identities.

The algebra (D, f) of Example 1.1 can be easily rewritten to obtain "almost i-zero" ternary 3 element algebra. More generally, similar examples of almost i-zero algebras can be produced from n-element i-trivial algebras with one n-ary operation. All such algebras are modes and none of them satisfies Szendrei identities. Consequently, they do not embed into semimodules over commutative semirings. To avoid technical complications, next sections deal only with modes with one ternary operation that generalize Example 1.1. However, it would be very easy to extend all the following notions and results to modes with one n-ary operation for all $n \geq 4$.

2. Ternary differential modes and differential groupoids

Let us start with collecting a couple of further remarks concerning the algebra (D, f).

The algebra (D, f) contains the two element left-zero subalgebra $\{0, 2\}$ and has the two element left-zero quotient. It follows that the variety $\mathcal{V}(D)$ generated by (D, f) contains a non-trivial subvariety (generated by this two element left-zero algebra) of Szendrei modes.

Lemma 2.1. *The algebra (D, f) satisfies the following identities:*

$$((xy_1y_2)z_1z_2) = ((xz_1z_2)y_1y_2) \qquad (left\,normal\,law), \tag{2.1}$$

and

$$(x(y_1z_1z_2)y_2) = (xy_1y_2) = (xy_1(y_2t_1t_2)). \tag{2.2}$$

We omit an easy proof. Note that the last identities of Lemma 2.1 are equivalent to the following one:

$$(x(y_1z_1z_2)(y_2t_1t_2)) = (xy_1y_2) \qquad (\textit{left reductive law}). \tag{2.3}$$

Consider now the variety $\mathcal{D}_3$ of modes with one ternary operation f, defined by the identities (2.1) and (2.3). Call this variety the variety of *ternary differential modes*. The variety forms a ternary counterpart of the variety $\mathcal{D}_2$ of differential groupoids. Recall that the variety $\mathcal{D}_2$ of *differential groupoid modes* or briefly *differential groupoids* is the variety of groupoid modes defined by the identity

$$x \cdot xy = x,$$

or equivalently the idempotent law, the *left normal* law

$$xy \cdot z = xz \cdot y,$$

and the *reductive* law

$$x \cdot yz = x \cdot y.$$

See [10] (and in particular explanation concerning relations of differential groupoids and differential groups), and also [7], [8] and [12].

The identities (2.1) and (2.3) are counterparts of the left normal and reductive identities of differential groupoids. Now it seems quite obvious that most of the basic properties of differential groupoids carry over to their ternary counterpart. First easy observation shows that the varieties $\mathcal{D}_2$ and $\mathcal{D}_3$ have similar types of axiomatizations.

Proposition 2.2. *The variety $\mathcal{D}_3$ may be defined by any one of the three following sets of identities:*

1. *the idempotent, left normal and left reductive laws,*
2. *the idempotent, entropic and left reductive laws,*
3. *the idempotent and entropic laws and the following absorption law*

$$(x(xy_1z_1)(xy_2z_2)) = x \tag{2.4}$$

Proof. Let us show that the last set of identities implies the second one. Using first (2.4), then entropic law and finally (2.4) again, we obtain the following:

$$\begin{aligned}(xy_1y_2) &= ((xy_1y_2)((xy_1y_2)z_1t_1)((xy_1y_2)z_2t_2))\\ &= ((x(xy_1y_2)(xy_1y_2))(y_1z_1z_2)(y_2t_1t_2))\\ &= (x(y_1z_1z_2)(y_2t_1t_2)).\end{aligned}$$

The remaining implications are shown in a similar way. □

Now note that each word $x \circ y$ on two variables x and y, and with the left-most variable x, is equivalent in $\mathcal{D}_3$ to one of the following

$$xR^k_{xy}R^l_{yx}R^m_{yy}. \tag{2.5}$$

Here the symbol xR^n_{ab} means $((\dots((xab)ab)\dots)ab)$ with ab repeated n times, and $xR^0_{ab} = x$. (The meaning of the symbol R_{ab} will be explained in more details in Section 3.) We omit an easy inductive proof of this fact that uses left reductive, left normal and idempotent laws.

Proposition 2.3. *The binary derived operations $x \circ y$ of a ternary differential mode (A, f) determined by any of (2.5) is a differential groupoid operation.*

Proof. As all derived operations of a mode satisfy idempotent and entropic laws (see [12, Corollary 5.5]), it is sufficient to check that these operations satisfy the (binary) reductive law. Indeed, the left reductive law (2.3) implies that

$$x \circ (y \circ z) = xR^k_{x(y\circ z)}R^l_{(y\circ z)x}R^m_{(y\circ z)(y\circ z)} = xR^k_{xy}R^l_{yx}R^m_{yy} = x \circ y.$$

□

Note also that the binary derived operations with the left-most variable y are right differential groupoid operations.

It is well known that each derived operation of a differential groupoid $(G, \cdot)$ has the standard form

$$x_1x_2^{k_2}\dots x_n^{k_n} := (\dots(\dots((\dots(x_1 \underbrace{x_2)\dots)x_2}_{k_2-times})\dots)\underbrace{x_n\dots)x_n}_{k_n-times}.$$

In particular, each ternary derived operation can be written as

$$x_1x_2^{k_2}x_3^{k_3}.$$

Note also that together with the left reductive law, the Szendrei identities reduce in ternary differential modes to the following ones:

$$((x_{11}X_{12}x_{13})X_{21}x_{31}) = ((x_{11}X_{21}x_{13})X_{12}x_{31}), \tag{2.6}$$

$$((x_{11}x_{12}X_{13})x_{21}X_{31}) = ((x_{11}x_{12}X_{31})x_{21}X_{13}). \tag{2.7}$$

By the left normal law, they are equivalent.

Proposition 2.4. *Let $(G, \cdot)$ be a differential groupoid. Each ternary derived operation $(x_1x_2x_3) := x_1x_2^{k_2}x_3^{k_3}$ defines a ternary differential mode. Moreover, $(G,(x_1x_2x_3))$ satisfies the Szendrei identities.*

Proof. The operation $(x_1x_2x_3)$ is obviously idempotent and entropic. Let us check that it satisfies the (ternary) left reductive law. By the reductive law for differential groupoids one obtains the following:

$$(x(y_1z_1z_2)(y_2t_1t_2)) = x(y_1z_1^{k_2}z_2^{k_3})^{k_2}(y_2t_1^{k_2}t_2^{k_3})^{k_3} = xy_1^{k_2}y_2^{k_3} = (xy_1y_2).$$

Since $(G, \cdot)$ is a subreduct of a semimodule over a commutative semiring, so is its ternary reduct $(G,(x_1x_2x_3))$. Thus it satisfies the Szendrei identities. □

Lemma 2.5. *Each of the (equivalent) Szendrei identities* (2.6) *and* (2.7) *is equivalent, in $\mathcal{D}_3$, to the identity*

$$(xyz) = ((xyx)xz) \tag{2.8}$$

Proof. The new identity is an obvious consequence of the Szendrei identity (2.6) and the idempotent law. To show the reverse implication, we use the new identity, the left reductivity (2.3) and the left normality (2.1) to obtain the following:

$$\begin{aligned}
((xyz)uv) &= (((xyz)u(xyz))(xyz)v) && \text{(by (2.8))}\\
&= (((xyz)ux)xv) && \text{(by (2.3))}\\
&= ((((xyx)xz)ux)xv) && \text{(by (2.8))}\\
&= ((((xux)xz)yx)xv) && \text{(by (2.1))}\\
&= (((xuz)yx)xv) && \text{(by (2.8))}\\
&= (((xyx)xv)uz) && \text{(by (2.1))}\\
&= ((xyv)uz) && \text{(by (2.8))}\\
&= ((xuz)yv) && \text{(by (2.1))}
\end{aligned}$$

The second equivalence is proved in a similar way. □

3. Constructing ternary differential modes

It is well known that the variety $\mathcal{D}_2$ of differential groupoids coincides with the Mal'cev power $\mathcal{LZ} \circ \mathcal{LZ}$ of the variety of left-zero bands relative to the variety of groupoid modes. (See [12, Theorem 5.6.3].) In particular, this means that each differential groupoid has a left-zero semigroup as a homomorphic image with left-zero semigroups as blocks of the corresponding kernel. This gives a good basis for some structure theorems. We can expect that a similar situation will appear in the case of ternary differential modes.

In what follows we will use the name of a differential mode to denote a ternary differential mode, while reserving the name of differential groupoids for binary differential modes.

First we will describe a certain construction of differential modes, similar in spirit to a construction known for differential groupoids (see [7]). Let I be a non-empty set, and let A_i, where $i \in I$, be a family of non-empty sets. For each triple $(i, j, k) \in I^3$, let $h_{i,jk} : A_i \to A_i$ be a mapping such that

(a) $h_{i,ii}$ is the identity mapping on A_i,
(b) $h_{i,jk}h_{i,mn} = h_{i,mn}h_{i,jk}$.

Define a ternary operation f on the disjoint union $A := \bigcup_{i\in I} A_i$ by $(a_ib_jc_k) := a_ih_{i,jk}$, where $a_i \in A_i$, $b_j \in A_j$, $c_k \in A_k$. Here and frequently later on, we follow a familiar convention of denoting elements in a summand A_i of a sum A by small letters with the same index.

It is easy to see that each A_i is a subalgebra of (A, f), and is a left-zero algebra. Moreover, the mapping $A \to I; a_i \mapsto i$ is a homomorphism onto the left-zero algebra (I, f). One routinely checks, similarly as in the case of differential groupoids, that such a *sum* (A, f) *of left-zero algebras* (A_i, f) *over the left-zero algebra* (I, f) or briefly, an $LZ \circ LZ$*-sum* of (A_i, f) is a differential mode.

Next we will show that each differential mode has a left-zero quotient with corresponding left-zero congruence classes, such that it can be reconstructed from this quotient and the congruence classes as an $LZ \circ LZ$-sum. To provide appropriate decompositions of differential modes, we will first introduce several congruence relations.

For each pair (b, c) of elements of a differential mode (A, f), consider the right translation

$$R_{bc} : A \to A; x \mapsto (xbc).$$

The set $AR = \{R_{bc} \mid b, c \in A\}$ of right translations generates a submonoid $R(A)$ of the endomorphism monoid $End(A, f)$ of the differential mode. This monoid is called the *right mapping monoid* of the differential mode. By left normality, it is a commutative monoid. For an element a of A, the set

$$aR(A) := \{a\varphi \mid \varphi \in R(A)\}$$

is called the *orbit* of a in A.

Lemma 3.1. *Let a be an element of a differential mode (A, f). Then the orbit $aR(A)$ of a is a subalgebra of (A, f) and is a left-zero algebra.*

Proof. Let $b, c, d \in aR(A)$. Then there are $\varphi, \chi, \psi \in R(A)$ such that $b = a\varphi, c = a\chi, d = a\psi$. It follows by the left reductive and then by the left normal and idempotent laws that $(bcd) = (a\varphi\, a\chi\, a\psi) = (a\varphi\, a\, a) = a\varphi = b$. Hence indeed, the orbit $aR(A)$ is a subalgebra and a left-zero algebra. □

Two elements a and b of A are said to be in the relation β if the intersection of their orbits is non-empty:

$$a\,\beta\, b :\Leftrightarrow \exists c \in aR(A) \cap bR(A). \tag{3.1}$$

Note that for each $a \in A$ and $\varphi \in R(A)$

$$(a, a\varphi) \in \beta, \tag{3.2}$$

whence the orbit of a is contained in the β-class of a.

The following theorem is modeled on [10, Theorem 2.6]. Its proof shows typical similarities and differences between the binary and ternary cases.

Theorem 3.2. *The relation β of* (3.1) *is a congruence relation on (A, f). Moreover, it is the least congruence on (A, f) such that the quotient $(A/\beta, f)$ is a left-zero algebra.*

Proof. It follows by the definition that β is reflexive and symmetric. To show that it is transitive let $x, y, z \in A$ and $x\,\beta\, y\,\beta\, z$. This means that there are $\upsilon, \varphi, \chi, \psi \in R(A)$ such that $x\upsilon = y\varphi$ and $y\chi = z\psi$. Then $x\upsilon\chi = y\varphi\chi = y\chi\varphi = z\psi\varphi$, so that $x\,\beta\, z$, and β is transitive. Now (3.2) implies that for all $a, b \in A$, one has $x\,\beta\,(xab)$. Thus for any $a, b, c, d \in A$, and in particular for $a\,\beta\, c$ and $b\,\beta\, d$, if $x\,\beta\, y$, then

$$(xab)\,\beta\, x\,\beta\, y\,\beta\,(ycd).$$

Hence $(xab)\,\beta\,(ycd)$ by the transitivity, so that β is a congruence relation on (A, f). Moreover, as $(x/\beta\, a/\beta\, b/\beta) = (xab)/\beta = x/\beta$, the quotient $(A/\beta, f)$ is a left-zero algebra. Finally, suppose that $(A/\alpha, f)$ is a left-zero algebra. Recall that $x\upsilon = y\varphi$ for some υ, $\varphi \in R(A)$. Set $\upsilon = R_{a_1b_1} \dots R_{a_mb_m}$ and $\varphi = R_{c_1d_1} \dots R_{c_nd_n}$. Then

$$x/\alpha = (x/\alpha)R_{(a_1/\alpha)\,(b_1/\alpha)} \dots R_{(a_m/\alpha)\,(b_m/\alpha)} = (x\upsilon)/\alpha$$

$$= (y\varphi)/\alpha = (y/\alpha)R_{(c_1/\alpha)\,(d_1/\alpha)} \dots R_{(c_n/\alpha)\,(d_n/\alpha)} = y/\alpha,$$

whence $\beta \leq \alpha$. □

Note that the largest left-zero quotient of (A, f) (obtained by the least left-zero congruence) is the *left-zero replica* of (A, f). (See [12, Section 3.3].)

We will define two more relations γ and δ on a differential mode (A, f). For $a, b \in A$ set

$$(a, b) \in \gamma :\Leftrightarrow \forall x, y \in A,\ (xya) = (xyb).$$

And similarly, for $a, b \in A$ set

$$(a, b) \in \delta :\Leftrightarrow \forall x, y \in A,\ (xay) = (xby).$$

Lemma 3.3. *The relations γ and δ are both congruence relations on (A, f). Moreover the quotients $(A/\gamma, f)$ and $(A/\delta, f)$ are left-zero algebras.*

Proof. We prove that γ satisfies the conditions of the lemma. The proof for δ is similar. The relation γ is obviously an equivalence relation. To show that it is a congruence relation, let $a_i, b_i \in A$ and $a_i\,\gamma\, b_i$ for $i = 1, 2, 3$. This means that for all $x, y \in A$ one has $(xya_i) = (xyb_i)$. Hence by the left reductive law

$$(xy(a_1a_2a_3)) = (xya_1) = (xyb_1) = (xy(b_1b_2b_3)).$$

It follows that γ is a congruence relation. Moreover $(xy(abc)) = (xya)$ for any $x, y, a, b, c \in A$, whence $((abc), a) \in \gamma$, and the quotient $(A/\gamma, f)$ is a left-zero algebra. $\square$

Now define $\alpha := \delta \cap \gamma$.

Lemma 3.4. *Both the quotient $(A/\alpha, f)$ and the corresponding congruence classes are left-zero algebras.*

Proof. By Lemma 3.3, the congruence β is contained both in γ and in δ, and hence also in their intersection α. Thus the quotient (A/α) is a left-zero algebra.

Now let $a \ \alpha \ b \ \alpha \ c$. As $b \, \gamma \, c$, the definition of γ implies that for $x = a$ and $y = b$, one has $(abb) = (abc)$. Similarly, since $a \, \gamma \, b$, one obtains for $x = y = a$ that $a = (aab)$. Finally, as $a \, \delta \, b$, the definition of δ implies that for $x = a$ and $y = b$, one has $(aab) = (abb)$. Together this shows that $a = (aab) = (abb) = (abc)$, whence the α-class of a is a left-zero algebra. $\square$

Let us note that since the relation β is the smallest congruence on (A, f) with the left-zero quotient, it follows that $\beta \leq \alpha$, and since all α-classes are left-zero algebras, also all β-classes have to be left-zero algebras.

Theorem 3.5. *Let θ be a congruence of a differential mode (A, f) such that $\beta \leq \theta \leq \alpha$. Then θ provides a decomposition of (A, f) into an $LZ \circ LZ$-sum of left-zero θ-classes over the left-zero quotient $(A/\theta, f)$.*

Proof. Let $I := A/\theta$ and let A_i be the θ-classes of (A, f). Then both the quotient and the corresponding congruence classes are left-zero algebras. Let $b \ \theta \ b'$ and $c \ \theta \ c'$. Then obviously $b \ (\delta \cap \gamma) \ b'$ and $c \ (\delta \cap \gamma) \ c'$. The definition of δ shows that for $x = a$ and $y = c$, one has $(abc) = (ab'c)$. And then the definition of γ shows that for $x = a$ and $y = b'$, one has $(ab'c) = (ab'c')$. Hence $(abc) = (ab'c')$. Now for each triple $(i, j, k) \in I^3$, $a_i \in A_i$ and any $b_j \in A_j$, $c_k \in A_k$ define

$$h_{i,jk} : A_i \to A_i; a_i \mapsto a_i h_{i,jk} =: (a_i b_j c_k).$$

Then clearly

$$h_{i,ii} : A_i \to A_i; a_i \mapsto a_i h_{i,ii} = (a_i b_i c_i) = a_i.$$

Moreover, the left normal law implies that for any $d_m \in A_m$ and $e_n \in A_n$

$$a_i h_{i,jk} h_{i,mn} = ((a_i b_j c_k) d_m e_n) = ((a_i d_m e_n) b_j c_k) = a_i h_{i,mn} h_{i,jk}.$$

It follows that (A, f) is the $LZ \circ LZ$-sum of the left-zero algebras (A_i, f) over the left-zero algebra (I, f). $\square$

Theorem 3.5 shows that, similarly as in the binary case, a ternary differential mode (A, f) may have many $LZ \circ LZ$-congruence relations decomposing it into an $LZ \circ LZ$-sum. Let us note, that the congruence α is the greatest such congruence relation. To show this, let θ be an $LZ \circ LZ$-congruence on (A, f) with $I := A/\theta$ and the θ-classes A_i, for $i \in I$. Consider $a, b \in A$ with $(a, b) \in \theta$. Then $a, b \in A_k$ for some θ-class A_k, with $k \in I$. We verify that $(a, b) \in \gamma$. Let $x \in A_i$ and $y \in A_j$. Then $(xya) = xh_{i,jk}$. Since $b \in A_k$, we obtain $(xyb) = xh_{i,jk} = (xya)$. Similarly we verify that $(a, b) \in \delta$. Hence, $(a, b) \in \gamma \cap \delta = \alpha$. It follows that all congruence relations of a differential mode (A, f) providing an $LZ \circ LZ$-sum decomposition contain β and are contained in α.

Example 3.6. The algebra (D, f) of Example 1.1 decomposes as the $LZ \circ LZ$-sum of two subalgebras $D_0 = \{0, 2\}$ and $D_1 = \{1\}$ with $xh_{0,11} = 2 - x$ and otherwise $xh_{i,jk} = x$.

Example 3.7. A differential mode may have a congruence with left-zero quotient and left-zero congruence classes that do not provide a decomposition into an $LZ \circ LZ$-sum. Let $A = \{0, 1, 2, 3\}$. Define a ternary operation f on A by $(312) = 0$, $(012) = 3$ and otherwise $(xyz) = x$. It is easy to check that (A, f) is a differential mode, and that the relations α and β coincide, and have three congruence classes $\{0, 3\}, \{1\}, \{2\}$. The relation γ has two classes $A_3 = \{0, 1, 3\}$ and $A_2 = \{2\}$ and the relation δ also has two classes $A_0 = \{0, 2, 3\}$ and $A_1 = \{1\}$. Both have left-zero quotients and left-zero classes. But none of them provides an $LZ \circ LZ$-sum. Indeed, if say δ would provide such a decomposition, then we should have $x_0h_{0,10} = (x_0y_1z_0)$ for any choice of z_0 in A_0. However $(012) = 3$ but $(013) = 0$.

Example 3.8. Though the orbit of each element of a differential mode is always contained in one β-class, the relation between orbits and β-classes may be quite complicated. Consider the following example. Let A be the disjoint sum of the set $\mathbb{R}$ of real numbers and one element set $\{\infty\}$. Define the ternary operation f on A by setting

$$(abc) := \begin{cases} a+1, & \text{if } a \in \mathbb{R} \text{ and } (b = \infty \text{ or } c = \infty); \\ a, & \text{otherwise.} \end{cases}$$

It is easy to check that (A, f) is a differential mode with two α-classes $\mathbb{R}$ and $\{\infty\}$. For any $a \in \mathbb{R}$, the orbit $aR(A)$ consists of all numbers $a+n$ for positive integers n. The relation β may be described as follows. First $\infty/\beta = \{\infty\}$. Then for any $a, b \in \mathbb{R}$,

$$a \,\beta\, b \;\Leftrightarrow\; a - b \in \mathbb{Z}.$$

It follows that for each $a \in \mathbb{R}$, one has $a/\beta = \{a + c \mid c \in \mathbb{Z}\}$ and $aR(A) \subset a/\beta$. The quotient set $\mathbb{R}/\beta$ coincides with the underlying set of the quotient group $\mathbb{R}/\mathbb{Z}$.

Example 3.9. A differential mode (A, f) is called *k-cyclic* if it satisfies the *k-cyclic law*

$$xR^k_{yz} = x, \tag{3.3}$$

for some $k \in \mathbb{N}$. For any two elements a and b of such a mode, the orbits $aR(A)$ and $bR(A)$ either coincide or are disjoint, and the β-classes coincide with the orbits of (A, f). An example of an k-cyclic differential mode is given by the algebra $(\mathbb{Z}_{k^2}, \underline{k})$, where

$$\underline{k}(a, b, c) := a - bk + ck.$$

Example 3.10. A finite differential mode (A, f), with β-classes A_i for $i \in I$, can be represented by a labeled directed graph with elements of A as vertices, and edges labeled by pairs $(j, k) \in I^2$. There is an edge from $b_i \in A_i$ to $c_i \in A_i$ labeled by (j, k) if for any $x_j \in A_j$ and any $y_k \in A_k$, $c_i = (b_i x_j y_k) = b_i h_{i,jk}$. The β-classes provide connected components of the graph. Each element of a β-class is an initial point of precisely $|I \times I|$ edges. See [5] for a similar representation of differential groupoids.

Let $\mathcal{LZ}_3$ be the variety of ternary left-zero algebras. Let $\mathcal{LZ}_3 \circ \mathcal{LZ}_3$ be the Mal'cev product of $\mathcal{LZ}_3$ and $\mathcal{LZ}_3$ relative to the variety of all modes with one ternary operation. (See [12, Section 3.7].)

Corollary 3.11. *The variety $\mathcal{D}_3$ of differential modes coincides with the Mal'cev power $\mathcal{LZ}_3 \circ \mathcal{LZ}_3$.*

Proof. By Theorem 3.5, $\mathcal{D}_3 \subseteq \mathcal{LZ}_3 \circ \mathcal{LZ}_3$. Now assume that a mode (A, f) is in $\mathcal{LZ}_3 \circ \mathcal{LZ}_3$. There is a congruence θ of (A, f) such that $(A/\theta, f)$ is in $\mathcal{LZ}_3$ and for each $a \in A$, the subalgebra $(a/\theta, f)$ is also in $\mathcal{LZ}_3$. The first statement implies that $(a/\theta\ b/\theta\ c/\theta) = (abc)/\theta = a/\theta$ for any $a, b, c \in A$, whence the elements $a, (abc)$ and $(ab'c')$ are in the class a/θ. As $(a/\theta, f)$ is a left-zero algebra, it follows that $(a(abc)(ab_1c_1)) = a$. Hence the absorption law holds in (A, f), and (A, f) is a differential mode. □

4. Free ternary differential modes

First we show that an identity satisfied in a non-trivial differential mode must have the same leftmost variables.

Lemma 4.1. *If a differential mode satisfies an identity with different leftmost variables, then it is trivial.*

Proof. As each differential mode is an $LZ \circ LZ$-sum of left-zero subalgebras, it follows that the variety $\mathcal{LZ}_3$ is the only non-trivial minimal subvariety of the variety $\mathcal{D}_3$. Now an identity holds in the variety $\mathcal{LZ}_3$ precisely if its leftmost variables are equal. Hence an identity satisfied by a differential mode must have leftmost variables equal and a differential mode satisfying an identity with different leftmost variables must be trivial. □

Let $\mathbf{n} := \{1, \ldots, n\}$. Consider the cartesian product $\mathbf{n} \times \mathbf{n}$ and denote its elements just as strings ij without commas and parentheses. Let X be a set of variables, and $F(X) := F_{\mathcal{D}_3}(X)$ be the free $\mathcal{D}_3$-algebra over X. We will identify elements of $F(X)$ with words (terms) representing them.

For $x_i, x_j \in X$, denote the right translation $R_{x_i x_j} : F(X) \to F(X)$ by R_{ij}. We will frequently use the "right translation" notation when writing words and identities. This allows us to reduce the number of parentheses, and clearly shows the structure of words and of algebras being constructed.

Theorem 4.2. *If $w = w(x_1, \ldots, x_n)$ is an element of the free $\mathcal{D}_3$-algebra $F(X)$ over a set X, the set $\{x_1, \ldots, x_n\}$ is precisely the set of variables in w and x_1 is its leftmost variable, then w may be expressed in the standard form*

$$x_1 R_{12}^{k_{12}} \ldots R_{1n}^{k_{1n}} R_{21}^{k_{21}} \ldots R_{2n}^{k_{2n}} \ldots R_{n1}^{k_{n1}} \ldots R_{nn}^{k_{nn}}, \tag{4.1}$$

where the indices ij run over the set $\mathbf{n} \times \mathbf{n}$ and are ordered lexicographically. The algebra $F(X)$ is the $LZ \circ LZ$-sum of the orbits of its generators in X.

Note that $k_{ij} = 0$ is possible, and in this case $yR_{ij}^{k_{ij}} = y$.

Proof. We omit a standard inductive proof of the first part, showing that w has the required form. It is similar to the binary case, and follows directly by the defining identities of differential modes. By Lemma 4.1, the orbits of any two generators are disjoint. The last statement of the theorem follows by the fact that the decomposition of $F(X)$ into the union of orbits of generators in X provides an $LZ \circ LZ$ decomposition of $F(X)$. The mappings $h_{i,jk} : A_i \to A_i$, where A_i is the orbit of x_i, defined by $a_i \mapsto (a_i x_j x_k)$, satisfies the defining conditions of the $LZ \circ LZ$-sum. Indeed, the operation f applied to three elements of one orbit provides always the leftmost element. When applied to elements w_i, w_j and w_k of the orbits of x_i, x_j and x_k, respectively, the results is $(w_i x_j x_k)$, and it may be easily reduced to the form of (4.1). In particular, this shows the uniqueness of the standard form. □

For a variety $\mathcal{V}$ of differential modes, let us call the subvariety $Sz(\mathcal{V})$, defined by any of the Szendrei identities (2.6) and (2.7), the *Szendrei subvariety* of $\mathcal{V}$, and its members *Szendrei modes*. In particular, $Sz(\mathcal{D}_3)$ is the subvariety of all Szendrei differential modes. Next theorem will describe free Szendrei differential modes. First we present a technical lemma that will help with subsequent calculations.

Lemma 4.3. *In each differential mode, the Szendrei identities imply the following*

(a) $xR^k_{yz}R^{k+i}_{zt} = xR^k_{yt}R^k_{zz}R^i_{zt}$;
(b) $xR^{k+i}_{yz}R^k_{zt} = xR^k_{yt}R^k_{zz}R^i_{yz}$;
(c) $x_1R^k_{12}\dots R^k_{1n}R^k_{21}\dots R^k_{2n}\,\dots\,R^k_{n1}\dots R^k_{nn} = x_1R^{kn}_{22}\dots R^{kn}_{nn}$;
(d) $xR^k_{12}R^k_{23}\dots R^k_{n-1n} = xR^k_{22}\dots R^k_{n-1n-1}R^k_{1n}$,

for all positive integers k and natural i.

Proof. Applying the Szendrei identity of (2.6) and the left normal law, one gets

$$xR^k_{yz}R^{k+i}_{zt} = xR^k_{yz}R^k_{zt}R^i_{zt} = xR^k_{yt}R^k_{zz}R^i_{zt}.$$

This proves (a), and (b) is proved in a similar way. Easy inductive proofs of (c) and (d) are left to the readers. □

Theorem 4.4. *If $w = w(x_1,\dots,x_n)$ is an element of the free $Sz(\mathcal{D}_3)$-algebra $F_{Sz}(X)$ over a set X, the set $\{x = x_1,\dots,x_n\}$ is precisely the set of variables in w and x is its leftmost variable, then w may be expressed in the standard form*

$$xR^{k_{11}}_{i_1j_1}R^{k_{22}}_{i_2j_2}\dots R^{k_{ss}}_{i_sj_s}, \tag{4.2}$$

where $i_p, j_p \in \{1,\dots,n\}$, for each $p = 1,\dots,s$,

$$i_1 \le \dots \le i_s \text{ and } j_1 \le \dots \le j_s.$$

Moreover $(x_{i_p}, x_{j_p}) \neq (x_{i_q}, x_{j_q})$ for $p \neq q$, and

$$x \notin \{x_{i_1},\dots,x_{i_s}\} \cap \{x_{j_1},\dots,x_{j_s}\}.$$

The algebra $F_{Sz}(X)$ is the $LZ \circ LZ$-sum of the orbits of its generators in X.

Proof. By Theorem 4.2, each element w of $F_{Sz}(X)$, can be presented in the form (4.1).

We show that in the variety of Szendrei modes, each such term can be reduced to the form (4.2). First note that there is one term x in the standard form with one variable x. Then, by Lemma 4.3, all terms with two variables x_1 and x_2 can be reduced to one of the following:

$$xR^{k_{12}}_{12}, \quad xR^{k_{21}}_{21}, \quad xR^{k_{22}}_{22}, \quad xR^{k_{12}}_{12}R^{k_{22}}_{22} \quad \text{and} \quad xR^{k_{21}}_{21}R^{k_{22}}_{22}.$$

(Note that $xR_{12}^{k_{12}}R_{21}^{k_{21}}$ equals $xR_{12}^{k_{12}-k_{21}}R_{22}^{k_{21}}$ or $xR_{21}^{k_{21}-k_{12}}R_{22}^{k_{12}}$.) Then consider a general term in the form (4.1). Using Szendrei identities, we can reorder arbitrarily variables in the set $\{x_{i_1},\ldots,x_{i_{s+1}}\}$ and in the set $\{x_{j_1},\ldots,x_{j_{s+1}}\}$. Finally, if x appears among both the $\{x_{i_1},\ldots,x_{i_{s+1}}\}$ and $\{x_{j_1},\ldots,x_{j_{s+1}}\}$, then both occurrences are moved, using Szendrei identities, to the leftmost x, and then disappear by idempotency. □

Let us call a differential mode (A,f) *hemisemiprojection* if it satisfies the identities

$$(xxy) = (xyx) = x. \tag{4.3}$$

We will show that almost all modes in the variety $hs(\mathcal{D}_3)$ of hemisemiprojection differential modes are not Szendrei modes.

Proposition 4.5. *The Szendrei subvariety of the variety $hs(\mathcal{D}_3)$ coincides with the variety $\mathcal{LZ}_3$ of the left-zero algebras.*

Proof. By Lemma 2.5, the Szendrei identities are equivalent to the identity (2.8). Hence in each hemisemiprojection Szendrei mode,

$$(xyz) = ((xyx)xz) = (xxz) = x.$$

Obviously, $\mathcal{LZ}_3 \subseteq hs(\mathcal{D}_3)$. □

Proposition 4.5 allows us to find a new family of modes non-embeddable into semimodules over commutative semirings.

Corollary 4.6. *A hemisemiprojection mode embeds as a subreduct into a semimodule over a commutative semiring if and only if it is a left-zero algebra.*

Example 4.7. It can be easily checked that the basic algebra (D,f) of Example 1.1, not embeddable into a semimodule over a commutative semiring, is a hemisemiprojection mode, but satisfies also other identities, as e.g. $(xyz) = (xzy)$ and $((xyz)yz) = x$.

Proposition 4.8. *If $w = w(x,x_1,\ldots,x_n)$ is an element of the free $hs(\mathcal{D}_3)$-algebra $F_{hs}(X)$ over a set X, the set $\{x,x_1,\ldots,x_n\}$ is precisely the set of variables in w and x is its left-most variable, then w may be expressed in the standard form*

$$xR_{12}^{k_{12}}\ldots R_{1n}^{k_{1n}}\ldots R_{n1}^{k_{n1}}\ldots R_{nn-1}^{k_{nn-1}}, \tag{4.4}$$

where the indices ij run over the set $\mathbf{n}\times\mathbf{n}$ and are ordered lexicographically. The algebra $F_{hs}(X)$ is the $LZ\circ LZ$-sum of the orbits of its generators in X.

Note that, similarly as in the case of Theorem 4.2, $k_{ij}=0$ is possible, and in this case $yR_{ij}^{k_{ij}} = y$.

Proof. The proof follows directly by Theorem 4.2, the hemisemiprojection laws and left-normal law. □

A hemisemiprojection mode satisfying the identity

$$(xyy) = x$$

is called *semiprojection*. The variety of semiprojection modes is denoted by $sp(\mathcal{D}_3)$. As shown by I. Rosenberg [15] clones generated by an n-ary semiprojection operation belong to five types of minimal clones. In [3], K. Kearnes and A. Szendrei have shown that semiprojection modes are always differential.

Free (ternary) semiprojection modes can be characterized in a similar way as free hemisemiprojection modes.

Corollary 4.9. *If $w = w(x, x_1, \dots, x_n)$ is an element of the free $sp(\mathcal{D}_3)$-algebra $F_{sp}(X)$ over a set X, the set $\{x, x_1, \dots, x_n\}$ is precisely the set of variables in w and x is its left-most variable, then w may be expressed in the standard form*

$$xR_{12}^{k_{12}} \dots R_{1n}^{k_{1n}} \dots R_{n1}^{k_{n1}} \dots R_{nn-1}^{k_{nn-1}}, \tag{4.5}$$

where the indices ij run over the set $\mathbf{n} \times \mathbf{n} - \{22, 33, \dots, nn\}$ and are ordered lexicographically. The algebra $F_{sp}(X)$ is the $LZ \circ LZ$-sum of the orbits of its generators in X.

Example 4.10. An example of a non-Szendrei semiprojection mode is provided by the following $Lz \circ Lz$-sum. Let A_0 and $C = \{c_i \mid i \in I\}$ be two disjoint non-empty sets and let $A_i = \{c_i\}$ for $i \in I$. Assume that A_0 has at least two elements. Let $A = \bigcup_{i \in I} A_i$ be the $Lz \circ Lz$-sum of A_i by the mappings $h_{i,jk}$, where all mappings $h_{i,jk}$ with $i \neq 0$, are identity mappings, at least one of $h_{0,ij}$ is not the identity mapping, and the ternary operation f is defined by $(a_0 c_i c_j) = a_0 h_{0,ij}$ for $a_0 \in A_0$ and $(xyz) = x$ otherwise. It is easy to check that the algebra (A, f) is a non-Szendrei hemisemiprojection mode, and in the case all $h_{0,ii}$ are also identity mappings, it is a semiprojection mode. In particular, the differential mode of Example 3.7 belongs to this family.

5. Identities and varieties

The lattice of varieties of differential groupoids is well-known. (See [7].) Each proper non-trivial subvariety of $\mathcal{D}_2$ is defined by the axioms of $\mathcal{D}_2$ and additional unique identity of the form $xy^{i+j} = xy^i$ for some natural number i and positive integer j. These subvarieties form the lattice $\mathcal{L}(\mathcal{D}_2)^-$ isomorphic with the direct product of two lattices of natural numbers, one with the divisibility relation as an ordering relation and the other one with the usual linear ordering. We will show that the lattice of varieties of differential modes is much more complicated. Though proper non-trivial finitely based varieties can be defined also by one more additional identity, the number of variables in such identities grow rapidly, and there are varieties not having a finite basis for their identities.

Note that by Lemma 4.1, if an identity $t = w$ holds in a non-trivial differential mode, then both t and w have the same leftmost variable. Moreover, by Theorem 4.2, both t and w can be written in the standard form described in this theorem.

Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Denote a term

$$t = x_1 R_{11}^{k_{11}} R_{12}^{k_{12}} \ldots R_{1m}^{k_{1m}} \ldots R_{m1}^{k_{m1}} \ldots R_{mm}^{k_{mm}}$$

over X by $x_1 R_k(X)$, and similarly a term

$$w = y_1 R_{11}^{i_{11}} R_{12}^{i_{12}} \ldots R_{1n}^{i_{1n}} \ldots R_{n1}^{i_{n1}} \ldots R_{nn}^{i_{nn}}$$

over Y by $y_1 R_i(Y)$, where k denotes the sequence $(k_{11}, \ldots, k_{mm})$ and i has a similar meaning.

Lemma 5.1. *Let $x_1 = y_1 =: x$ and $X \cap Y = \{x\}$. Assume that $t_1 := xR_k(X)$, $t_2 := xR_l(X)$, $w_1 := xR_i(Y)$, $w_2 := xR_j(Y)$. Then the identities*

$$t_1 = xR_k(X) = xR_l(X) = t_2 \text{ and } w_1 = xR_i(Y) = xR_j(Y) = w_2$$

are satisfied in $\mathcal{D}_3$ if and only if the identity

$$xR_k(X)R_i(Y) = xR_l(X)R_j(Y).$$

holds in $\mathcal{D}_3$.

Proof. ($\Rightarrow$) Substitute $xR_k(X)$ for x in w_1 and $xR_l(X)$ for x in w_2. Left reductivity shows that at any non-leftmost occurrence of x, the word $xR_p(X)$ will then reduce to x, and finally we obtain $xR_k(X)R_i(Y) = xR_l(X)R_j(Y)$.

($\Leftarrow$) Now consider $xR_k(X)R_i(Y) = xR_l(X)R_j(Y)$, and first substitute x for all variables in $R_i(Y)$ and $R_j(Y)$. Then left normality and idempotency implies $t_1 = xR_k(X) = xR_l(X) = t_2$. Similarly, substitute x for all variables in $R_k(Y)$ and $R_l(Y)$ and deduce $w_1 = xR_i(Y) = xR_j(Y) = w_2$. □

Theorem 5.2. *Every proper subvariety of the variety $\mathcal{D}_3$ either has an equational basis consisting of the axioms of $\mathcal{D}_3$ and one additional identity, or is non-finitely based.*

Proof. This follows directly by Lemma 5.1, since without loss of generality we can always assume that in any two identities (like in Lemma

5.1) satisfied in a subvariety in question, $x_1 = y_1 =: x$ and $X \cap Y = \{x\}$. $\square$

Example 5.3. The variety $\mathsf{V}(\mathbf{D})$, generated by the algebra $\mathbf{D} = (D, f)$ of Examples 1.1 and 4.7, is relatively based by the identities $(xxy) = x$, $(xyz) = (xzy)$ and $((xyz)yz) = x$ of Example 4.7. This can be easily checked as follows. The identities above and Proposition 4.8 imply that the free $\mathsf{V}(\mathbf{D})$-algebra on two generators x and y consists of four elements: $x, y, (xyy)$ and (yxx), with two disjoint orbits $\{x, (xyy)\}$ and $\{y, (yxx)\}$. The two generated algebra $\mathbf{D}$ is obviously a homomorphic image of the free $\mathsf{V}(\mathbf{D})$-algebra on two generators, by the homomorphism defined as follows: $x \mapsto 0, (xyy) \mapsto 2$ and $y, (yxx) \mapsto 1$. Now the elements of the free $\mathsf{V}(\mathbf{D})$-algebra on a set $X = \{x_1, \dots x_n\}$, for $n > 1$, as in Proposition 4.8, may be expressed in the standard form

$$x_1 R_{22}^{k_{22}} \dots R_{2n}^{k_{2n}} \dots R_{n-1n-1}^{k_{n-1n-1}} R_{n-1n}^{k_{n-1n}} R_{nn}^{k_{nn}},$$

where each k_{ij} is 0 or 1. Observe that each such element is determined by the first variable and a set of unordered pairs of $X - \{x_1\}$. It is easy to check that different terms have different values in D. By Lemma 5.1 and Theorem 5.2, the variety $\mathsf{V}(\mathbf{D})$ can be defined by one additional identity

$$x_1 R_{12} R_{34} R_{56}^2 = x_1 R_{43}.$$

Let us note that when decreasing the number of identities defining a subvariety of the variety $\mathcal{D}_3$, the number of variables grows quickly.

Next theorem shows that indeed there are varieties of differential modes not having a finite basis for their identities. Before formulating this theorem let us describe certain family of differential modes that will play a crucial role in the proof.

Example 5.4. For a natural number n, let $B := \mathbf{2}^{n+1}$, where $\mathbf{2} := \{0, 1\}$, and let $C_0 := B \cup \{\infty\}$. Assume that $I = \{0, 1, \dots, n+1\}$ and let $C_i := \{c_i\}$, for $i = 1, \dots, n+1$, be a family of one-element sets, and $C := \{c_i \mid i = 1, \dots, n+1\}$. For each triple $(i, j, k) \in I^3$, we will define mappings $h_{i,jk} : C_i \to C_i$ satisfying the defining conditions of $LZ \circ LZ$-sums as follows. First define auxiliary functions $p_i : C_0 \to C_0$, for $i = 1, \dots, n+1$, by

$$\begin{aligned}
&(b_1, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_{n+1}) \mapsto (b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n+1}),\\
&(b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_{n+1}) \mapsto \infty,\\
&\infty \mapsto \infty.
\end{aligned}$$

Then in the case $n = 0$, put $h_{0,01} = p_1$, let $h_{0,10}$ and $h_{0,11}$ take C_0 to ∞, and let all the remaining maps be the identity mappings. For $n > 0$

put

$$h_{0,01} = h_{0,10} = p_1, \quad h_{0,nn+1} = p_{n+1},$$

and for $i = 1, \ldots, n-1$,

$$h_{0,ii+1} = h_{0,i+1i} = p_{i+1}.$$

Define $h_{i,ii}$ and all mappings $h_{i,jk}$ for $i \neq 0$ to be identity mappings, and all other mappings $h_{0,jk}$ to be constant mappings taking all elements of C_0 to ∞. It is easy to see that any two of these mappings commute. Note also that $h_{0,ij} = h_{0,ji}$ unless $\{i,j\} = \{n, n+1\}$, and that $p_i p_i$ is the constant mapping taking all elements of C_0 to ∞. A ternary operation $f = (xyz)$ is defined on the disjoint union $A_n := \bigcup_{i \in I} C_i$ of all C_i by

$$(x_i y_j z_k) = x_i h_{i,jk},$$

for $x_i \in C_i, y_j \in C_j, z_k \in C_k$. Note in particular that for $n > 0$,

$$(x_0 y_0 c_i) = (x_0 x_0 c_i) = (x_0 c_i x_0) = (x_0 c_i y_0),$$

$$(\infty y_j z_k) = \infty,$$

and in the cases $i = j = k$ or $i \neq 0$, $j \neq 0$ and $k \neq 0$,

$$(x_i y_j z_k) = x_i.$$

Then obviously, each algebra $\mathbf{A}_n := (A_n, f)$ is the $Lz \circ Lz$-sum of left-zero algebras (C_i, f) over the left-zero algebra (I, f), and hence is a differential mode. Note also that C_0 and C, as well as $C \cup \{\infty\}$ are subalgebras of $\mathbf{A}_n$ and are left-zero algebras, and that the set $\{\mathbf{1} := (1, \ldots, 1), c_1, \ldots, c_{n+1}\}$ is the unique minimal set of generators of the algebra $\mathbf{A}_n$.

Theorem 5.5. *Let $\mathcal{V}$ be the subvariety of the variety $\mathcal{D}_3$ defined by the identity*

$$(d_{2,3}) \qquad xR_{yz}^3 = xR_{yz}^2$$

and for natural numbers n all the identities

$$(e_n) \qquad xR_{01}R_{12}R_{23}\ldots R_{n-1n}R_{nn+1} = xR_{01}R_{12}R_{23}\ldots R_{n-1n}R_{n+1n},$$

where $x = x_0$ and R_{ij} is $R_{x_i x_j}$. The variety $\mathcal{V}$ is locally finite, but has no finite basis for its identities.

Proof. First note, that by Theorem 4.2 and $(d_{2,3})$, each term representing an element of a finitely generated free $\mathcal{V}$-algebra, can be reduced to the form of (4.1) with all k_{ij} not bigger than 2. It follows that such free algebras are finite and hence $\mathcal{V}$ is locally finite.

Next observe that the identity (e_{n+1}) implies the identity (e_n). Indeed, it is enough to substitute x for x_1 in (e_{n+1}) to obtain (e_n). It follows that if a differential mode does not satisfy (e_n) for some n, then it does not satisfy (e_m) for any $m \geq n$.

We will prove that the variety $\mathcal{V}$ has no equational basis with a finite number of variables. We will achieve this by first showing that for no n, the algebra $\mathbf{A}_n$ satisfies the identity (e_n) (which has $n+2$ variables), and then by showing that the proper subalgebras of each algebra $\mathbf{A}_n$, generated by $n+1$ elements, belong to $\mathcal{V}$, i.e. each $\mathbf{A}_n$ satisfies all the identities of $\mathcal{V}$ in $n+1$ variables. Note that all algebras $\mathbf{A}_n$ satisfy the identity $(d_{2,3})$.

We first show that the algebra $\mathbf{A}_n$ does not satisfy the identity (e_n). Indeed,

$$((...((\mathbf{11}c_1)c_1c_2)...)c_{n-1}c_n) = \mathbf{1}h_{0,01}h_{0,12}...h_{0,n-1n} = (0,\dots,0,1),$$

and hence

$$(((...((\mathbf{11}c_1)c_1c_2)...)c_{n-1}c_n)c_nc_{n+1}) = (0,\dots,0,1)h_{0,nn+1} = (0,\dots,0,0),$$

while

$$(((...((\mathbf{11}c_1)c_1c_2)...)c_{n-1}c_n)c_{n+1}c_n) = (0,\dots,0,1)h_{0,n+1n} = \infty.$$

Now we will prove that each algebra $\mathbf{A}_n$ satisfies all identities true in $\mathcal{V}$ involving $n+1$ variables, i.e. all identities (e_m) for $m < n$, and all consequences of (e_m) for $m \geq n$, with $n+1$ variables.

Note that each identity of $\mathcal{V}$ is satisfied by the subalgebras C_0 and $C \cup \infty$ of $\mathbf{A}_n$. We only need to check if the identities (e_m) are satisfied in any $(n+1)$-generated subalgebra of $\mathbf{A}_n$ in the case $x_0 = a_0 \in B$ and at least one of x_i, for $i > 0$, is c_i. For any such choice of elements of A_n, we obtain

$$l_m := a_0h_{0,0i_1}h_{0,i_1i_2}\dots h_{0,i_mi_{m+1}}, \tag{5.1}$$

where all i_k are in I, on the left-hand side of (e_m) and

$$r_m := a_0h_{0,0i_1}h_{0,i_1i_2}\dots h_{0,i_{m+1}i_m} \tag{5.2}$$

on the right-hand side.

Note that (e_0) is satisfied in all $\mathbf{A}_n$ for $n > 0$. And consider the smallest m such that the identity (e_m) fails in $\mathbf{A}_n$. If $i_1 = 0$, we go back to (e_{m-1}), which holds by our assumption. So assume that $i_1 \geq 1$. If $i_1 > 1$, then $l_m = r_m = \infty$, so assume that $i_1 = 1$. If $i_2 = 0$ or $i_2 \neq 2$, then again $l_m = r_m = \infty$. So we may assume that $i_2 = 2$. Continuing in the same way, we end up with the following

$$l_m := a_0h_{0,01}h_{0,12}\dots h_{0,mm+1}, \tag{5.3}$$

and

$$r_m := a_0h_{0,01}h_{0,12}\dots h_{0,m+1m}, \tag{5.4}$$

where any two indices $i, i+1$ are different.

First assume that $m = n-1$. Then as in this case $h_{0,n-1n} = h_{0,nn-1}$, it follows that $l_m = r_m$, and consequently (e_{n-1}) and all (e_m) for $m < n$ are satisfied, a contradiction with the assumption that (e_m) fails in $\mathbf{A}_n$.

Now assume that $m \geq n$. Left reductivity, $(d_{2,3})$ and the fact that $x_0 p_i p_i = \infty$ in $\mathbf{A}_n$, show that it is enough to consider all identities in $n+1$ variables resulting through identification of some variables in (e_m). In particular, it means that some of i_k in l_m and r_m must be equal. However, this is not possible, since any two indices $i, i+1$ in l_m and r_m are different and $m \geq n$.

It follows that the consequences of (e_m) with $n+1$ variables are satisfied in $\mathbf{A}_n$. $\square$

Remark 5.6. Note that each of the subvarieties $\mathsf{V}(\mathbf{A}_n) = \mathsf{HSP}(\mathbf{A}_n)$ for $n > 0$ contains the variety $\mathcal{V}$ of Theorem 5.5. It is easy to check that the algebras $\mathbf{A}_n$, for $n > 0$, do not satisfy the identity (2.8). Indeed, $(\mathbf{1}c_1c_2) = \mathbf{1}h_{0,12} = \mathbf{1}p_2 = (1, 0, 1, \ldots, 1)$, while $((\mathbf{1}c_1\mathbf{1})\mathbf{1}c_2) = \mathbf{1}h_{0,10}h_{0,02} = \infty$. In particular, for $n = 2$, the subalgebra generated by these three elements belongs to the variety $\mathcal{V}$. It follows, that none of the varieties $\mathcal{V}$ and $\mathsf{V}(\mathbf{A}_n)$ are Szendrei varieties.

Let us call a set of identities defining a subvariety $\mathcal{V}$ of $\mathcal{D}_3$ (like e.g. the set consisting of $(d_{2,3})$ and all (e_n)) a *relative basis* of $\mathcal{V}$.

Corollary 5.7. *There is no upper bound on the number of variables in relative bases of subvarieties of the variety $\mathcal{D}_3$.*

Consequently, one cannot hope for any convenient description of the whole lattice $\mathcal{L}(\mathcal{D}_3)$ of subvarieties of $\mathcal{D}_3$.

However, the lattice $\mathcal{L}(\mathcal{D}_3)$ contains sublattices isomorphic to the lattice of subvarieties of the variety $\mathcal{D}_2$, that are not difficult to trace.

Example 5.8. By Proposition 2.3, each of the derived binary operations of differential modes is in fact a differential groupoid operation. Consequently, each subvariety of $\mathcal{D}_3$ defined by one additional identity of the form $(xyz) = xR^k_{xy}R^l_{yx}R^m_{yy}$ (see (2.5) and Proposition 2.3), is equivalent to the variety of differential groupoids. (Similar reasoning would apply to an operation depending on the variables x and z.) The defining identities of the subvarieties of $\mathcal{D}_2$, when applied to the derived operation $x \circ y := xR^k_{xy}R^l_{yx}R^m_{yy}$, define also subvarieties of the variety $\mathcal{D}_3$, and provide a sublattice of the lattice $\mathcal{L}(\mathcal{D}_3)$ of varieties of differential modes, isomorphic to the lattice $\mathcal{L}(\mathcal{D}_2)$ of subvarieties of $\mathcal{D}_2$. For example the operation $(xyz) = x \circ y = ((xxy)yx)$ determines the subvarieties defined by the identities

$$xR^{i+j}_{xy}R^{i+j}_{yx} = xR^i_{xy}R^i_{yx}.$$

Remark 5.9. Similarly as in the case of (binary) differential groupoids, a (non-trivial ternary) differential mode is never equivalent to an affine semimodule over a commutative semiring (and in particular to an affine space over a commutative ring). If (A, f) would be equivalent to an affine semimodule, then one of the derived binary operations would be a commutative semigroup operation. This is not possible in non-trivial differential modes. In particular, a differential mode cannot have a semilattice derived operation. It follows also, that a non-trivial variety of differential modes is neither equivalent to a variety of affine spaces over a commutative ring, nor to a variety of semilattice modes.

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