ON AXIOMS OF BIQUANDLES

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Abstract. We prove that the two conditions from the definition of a biquandle by Fenn, Jordan-Santana, Kauffman [1] are equivalent and thus answer a question posed in the paper. We also construct a weak biquandle, which is not a biquandle.

According to Fenn, Jordan-Santana and Kauffman [1], biquandles provide powerful invariants of virtual knots and links. It is thus desirable to simplify their axioms as much as possible. The aim of this very short note is to answer two questions regarding the definition of biquandles raised in [1], Section 4. For a background, please consult [1] or [2].

A pair (X, S) is called a *switch*, if S is a permutation of X^2 such that

$$(\dagger)\qquad (S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S).$$

Put

$$S(x, y) = (x \circ y, y * x).$$

Originally, the notation in [1] was $S(x, y) = (y_x, x^y)$. We will use the infix notation in order to keep the computation below readable.

Now, apply the left side of $(\dagger)$ to a triple $(x, y, z) \in X^3$; the result is the triple $((x\circ y)\circ((y*x)\circ z), ((y*x)\circ z)*(x\circ y), z*(y*x))$. Similarly, the result of the right side on (x, y, z) is $(x \circ (y \circ z), ((y \circ z) * x) \circ (z * y), (z * y) * ((y \circ z) * x))$. Consequently, the identity $(\dagger)$ is equivalent to the following three *switch identities* in terms of the operations $\circ, *$:

$$x \circ (y \circ z) = (x \circ y) \circ ((y * x) \circ z) \tag{1}$$

$$x * (y * z) = (x * y) * ((y \circ x) * z) \tag{2}$$

$$((x * y) \circ z) * (y \circ x) = ((x \circ z) * y) \circ (z * x) \tag{3}$$

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A switch (X, S) is called *a birack*, if the mappings

$$L_z^\circ : x \mapsto z \circ x, \qquad L_z^* : x \mapsto z * x$$

are bijective for every $z \in X$. In that case, we define the operations of left division by

$$x \backslash_\circ y = (L_x^\circ)^{-1}(y) \quad \text{and} \quad x \backslash_* y = (L_x^*)^{-1}(y)$$

for every $x, y \in X$. Clearly, they satisfy the following *left division identities*:

$$x \circ (x \backslash_\circ y) = y, \qquad x \backslash_\circ (x \circ y) = y \tag{4}$$

$$x * (x \backslash_* y) = y, \qquad x \backslash_* (x * y) = y. \tag{5}$$

A birack satisfying the identities

$$(x \backslash_\circ x) \backslash_* (x \backslash_\circ x) = x \tag{6}$$

$$(x \backslash_* x) \backslash_\circ (x \backslash_* x) = x \tag{7}$$

is called a *biquandle*. The following lemma answers a question from [1].

Lemma. *The identities (6),(7) are equivalent in any birack (actually, in any algebra $(X, \circ, *, \backslash_\circ, \backslash_*)$ satisfying the identities (1), (2), (4), (5)).*

Proof. We prove that (1), (4), (5), (6) imply (7). (The other implication can be proved analogously with the role of $\circ, \backslash_\circ$ and $*, \backslash_*$ interchanged; particularly, (2) instead of (1) is necessary.)

First, note that (1) is equivalent to the identity

$$(x \circ y) \circ z = x \circ (y \circ ((y * x) \backslash_\circ z)). \tag{8}$$

Indeed, both identities say that $L_x^\circ L_y^\circ = L_{x \circ y}^\circ L_{y * x}^\circ$.

Let w stand for $(x \backslash_* x) \circ x$ and consider the following sequence of equalities:

$$\begin{aligned} w \circ x &= ((x \backslash_* x) \circ x) \circ x \\ &= (x \backslash_* x) \circ (x \circ ((x * (x \backslash_* x)) \backslash_\circ x)) \qquad \text{by (8)} \\ &= (x \backslash_* x) \circ (x \circ (x \backslash_\circ x)) \qquad \text{by (5)} \\ &= (x \backslash_* x) \circ x = w \qquad \text{by (4).} \end{aligned}$$

Dividing both sides on the left by w, we obtain

$$x = w \backslash_\circ w.$$

Substituting w for x in (6), we obtain

$$(w \backslash_\circ w) \backslash_* (w \backslash_\circ w) = w$$

and the last two identities together say that

$$x\backslash_* x = w = (x\backslash_* x) \circ x$$

Now, dividing both sides on the left by $x\backslash_* x$, we obtain (7). $\square$

The other question in [1] asks, if there is a weak biquandle, which is not a biquandle. A switch (X, S) is called *a weak birack*, if the mappings

$$L_z^\circ : x \mapsto z \circ x, \qquad\qquad L_z^* : x \mapsto z * x$$

are surjective for every $z \in X$. And a weak birack is called a *weak biquandle*, if for every x there exist y, z such that

$$y \circ x = y, x * y = x, z * x = z \text{ and } x \circ z = x$$

(we cannot use the identities (6), (7), because there is no (unique) division; it is proved in [1] that these two definitions are equivalent for biracks).

The answer is positive and here is an example. Consider the Prüfer group $\mathbb{Z}_{p^\infty}$ of p-adic integers, which is defined as the inverse limit of the cyclic groups $\mathbb{Z}_{p^k}$ with the obvious embeddings (or, equivalently, $\mathbb{Z}_{p^\infty}$ is isomorphic to the subgroup of $(\mathbb{Q}/\mathbb{Z}, +)$ consisting of all $\frac{a}{p^k} + \mathbb{Z}$, $0 \le a < p$, $k \ge 1$). The Prüfer p-group is well known to be divisible, in particular, the selfmapping $x \mapsto px = \underbrace{x + \cdots + x}_{p}$ of $\mathbb{Z}_{p^\infty}$ is surjective. However, $x \mapsto px$ is not injective, its kernel is $\mathbb{Z}_p$ (or, the elements $\frac{a}{p} + \mathbb{Z}$ in the other representation). Now, put

$$a \circ b = (1-p)a + pb \quad \text{ and } \quad a * b = b$$

for every $a, b \in \mathbb{Z}_{p^\infty}$. It is easy to check that $\mathbb{Z}_{p^\infty}$ with the operations $\circ, *$ forms a weak biquandle, but it is certainly not a biquandle.

The construction can be expressed more generally: instead of the abelian group $\mathbb{Z}_{p^\infty}$, we can use any module M over a ring R containing an element $p \in R$ such that the selfmapping $x \mapsto px$ of M is surjective, but not injective. Again, the set M with the operations $\circ, *$ forms a weak biquandle, which is not a biquandle.

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References

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[2] L. Kauffman, *Virtual knot theory,* European J. Comb. 20 (1999), 663-690.

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