Zadání

1. Vypočtěte limity:

   a) $\lim\limits_{x\to 1} \frac{x^2-1}{2x^2-x-1}$
   c) $\lim\limits_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$
   e) $\lim\limits_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1}$
   b) $\lim\limits_{x\to\infty} \left(\frac{x^3}{x^2+1}-x\right)$
   d) $\lim\limits_{x\to 2} \left(\frac{1}{x^2-2x}-\frac{x}{x^2-4}\right)$
   f) $\lim\limits_{x\to 1} \frac{x+x^2+x^3+\ldots+x^n-n}{x-1}$.

2. Vypočtěte limity:

   a) $\lim\limits_{x\to 0} \frac{(1+x)(1+2x)\ldots(1+nx)-1}{x}$
   b) $\lim\limits_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$
   c) $\lim\limits_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$.

3. Vypočtěte limity:

   a) $\lim\limits_{x\to\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right)$
   c) $\lim\limits_{x\to\infty} \sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$
   d) $\lim\limits_{x\to a_+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$.
   b) $\lim\limits_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

Řešení

1. a) $\lim\limits_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim\limits_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \lim\limits_{x\to 1} \frac{x+1}{2x+1} = \frac{2}{3}$.

   b) $\lim\limits_{x\to\infty} \left(\frac{x^3}{x^2+1} - x\right) = \lim\limits_{x\to\infty} \frac{x^3-x(x^2+1)}{x^2+1} = \lim\limits_{x\to\infty} \frac{-1}{x+\frac{1}{x}} = 0$.

   c)
$$\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} = \lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} \cdot \frac{\sqrt{9+2x}+5}{\sqrt{9+2x}+5} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt[3]{x}^2+2\sqrt[3]{x}+4} = \lim_{x\to 8} \frac{9+2x-25}{x-8} \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5}$$
$$= \lim_{x\to 8} 2 \cdot \frac{\sqrt[3]{x}^2+2\sqrt[3]{x}+4}{\sqrt{9+2x}+5} = \frac{12}{5}.$$

   d) $\lim\limits_{x\to 2} \left(\frac{1}{x^2-2x} - \frac{x}{x^2-4}\right) = \lim\limits_{x\to 2} \frac{(x+2)-x^2}{x(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{-(x-2)(x+1)}{x(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{-(x+1)}{x(x+2)} = \frac{-3}{8}$.

   e) $\lim\limits_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim\limits_{x\to 1} \frac{(x-1)(x^{99}+x^{98}+x^{97}+...+x-1}{x^{49}+x^{48}+...+x-1)} = \frac{98}{48} = \frac{49}{24}$.

   f)
$$\lim_{x\to 1} \frac{x+x^2+x^3+...+x^n-n}{x-1} = \lim_{x\to 1} \frac{x-1+x^2-1+x^3-1+...+x^n-1}{x-1}$$
$$= \lim_{x\to 1} \frac{(x-1)+(x-1)(x+1)+..+(x-1)(x^{n-1}+x^{n-2}+...+1)}{x-1}$$
$$= \lim_{x\to 1} \big(1+(x+1)+(x^2+x+1)+...+(x^{n-1}+x^{n-2}+...+1)\big)$$
$$= \lim_{x\to 1} \sum_{i=0}^{n-1}(n-i)x^i = \sum_{i=0}^{n-1}(n-i) = \frac{n(n+1)}{2}.$$

2. a) $\lim\limits_{x\to 0} \frac{(1+x)(1+2x)...(1+nx)-1}{x} = \lim\limits_{x\to 0} \frac{1+(x+2x+...+nx)+o(x)-1}{x} = 1+2+...+n = \frac{n(n+1)}{2}$.

   b)
$$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0} \frac{1+\binom{n}{1}mx+\binom{n}{2}m^2x^2+...+m^nx^n-\big(1+\binom{m}{1}nx+\binom{m}{2}n^2x^2+...+n^mx^n\big)}{x^2}$$
$$= \lim_{x\to 0} \frac{\binom{n}{2}m^2x^2-\binom{m}{2}n^2x^2+o(x^2)}{x^2} = \binom{n}{2}m^2 - \binom{m}{2}n^2 = \frac{mn}{2}(n-m).$$

   c)
$$\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x^2+2x-8))^{10}} = \lim_{x\to 2} \frac{((x-2)(x+1))^{20}}{((x-2)(x-2)(x+4))^{10}} = \lim_{x\to 2} \frac{(x+1)^{20}}{(x+4)^{10}}$$
$$= \frac{3^{20}}{6^{10}} = \frac{3^{10}}{2^{10}}.$$

3. a)
$$\lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right) = \lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right) \cdot \frac{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}{\sqrt[3]{x^2+1}^2+\sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1}+\sqrt[3]{x^2-1}^2}$$
$$= \lim_{x\to\infty} \frac{x^{\frac{4}{3}}}{x^{\frac{4}{3}}} \cdot \frac{\big(x^2+1-(x^2-1)\big)}{\sqrt[3]{1+\frac{1}{x^2}}^2+\sqrt[3]{1+\frac{1}{x^2}}\cdot\sqrt[3]{1-\frac{1}{x^2}}+\sqrt[3]{1-\frac{1}{x^2}}^2}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt[3]{1+\frac{1}{x^2}}^2+\sqrt[3]{1+\frac{1}{x^2}}\cdot\sqrt[3]{1-\frac{1}{x^2}}+\sqrt[3]{1-\frac{1}{x^2}}^2} = \frac{2}{3}.$$

b)

$$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}\cdot\frac{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3}{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3}\cdot\frac{\sqrt{x}+4}{\sqrt{x}+4}$$
$$=\lim_{x\to 16}\frac{x-16}{x-16}\cdot\frac{\sqrt{x}+4}{\sqrt[4]{x}^3+2\sqrt[4]{x}^2+2^2\sqrt[4]{x}+2^3}=\frac{4+4}{8+8+8+8}=\frac{1}{4}.$$

c)

$$\lim_{x\to\infty}\sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)=\lim_{x\to\infty}\sqrt{x^3}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)\cdot\frac{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}$$
$$=\lim_{x\to\infty}\sqrt{x^3}\cdot\frac{(\sqrt{x+1}+\sqrt{x-1})^2-4x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}$$
$$=\lim_{x\to\infty}\sqrt{x^3}\cdot\frac{x+1+2\sqrt{x+1}\sqrt{x-1}+x-1-4x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}$$
$$=\lim_{x\to\infty}\sqrt{x^3}\cdot\frac{2\sqrt{x+1}\sqrt{x-1}-2x}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\cdot\frac{2\sqrt{x+1}\sqrt{x-1}+2x}{2\sqrt{x+1}\sqrt{x-1}+2x}$$
$$=\lim_{x\to\infty}\sqrt{x^3}\cdot\frac{4(x+1)(x-1)-4x^2}{\sqrt{x+1}+\sqrt{x-1}+2\sqrt{x}}\cdot\frac{1}{2\sqrt{x+1}\sqrt{x-1}+2x}$$
$$=\lim_{x\to\infty}\frac{-4}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}+2}\cdot\frac{1}{2\sqrt{1+\frac{1}{x}}\sqrt{1-\frac{1}{x}}+2}=-\frac{1}{4}.$$

d)

$$\lim_{x\to a_+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\lim_{x\to a_+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}\cdot\frac{\sqrt{x}+\sqrt{a}+\sqrt{x-a}}{\sqrt{x}+\sqrt{a}+\sqrt{x-a}}=\lim_{x\to a_+}\frac{(\sqrt{x}+\sqrt{x-a})^2-a}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}$$
$$=\lim_{x\to a_+}\frac{(x+2\sqrt{x}\sqrt{x-a}+x-a)-a}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}=\lim_{x\to a_+}\frac{2((x-a)+\sqrt{x}\sqrt{x-a})}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}$$
$$=\lim_{x\to a_+}\frac{2(\sqrt{x-a}\sqrt{x-a}+\sqrt{x}\sqrt{x-a})}{\sqrt{(x-a)(x+a)}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}$$
$$=\lim_{x\to a_+}\frac{2(\sqrt{x-a}+\sqrt{x})}{\sqrt{(x+a)}(\sqrt{x}+\sqrt{a}+\sqrt{x-a})}=\frac{2\sqrt{a}}{\sqrt{2a}2\sqrt{a}}=\frac{1}{\sqrt{2a}}.$$

Důležité limity

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
- $\lim_{x\to 0} \frac{\mathrm{tg}\, x}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$
- $\lim_{x\to 1} \frac{\ln x}{x-1} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0} \frac{\mathrm{arctg}\, x}{x} = 1$
- $\lim_{x\to 1-} \frac{\arccos x}{\sqrt{1-x}} = \sqrt{2}$

Poznámka: výchozí limity, odvozené limity a rozšiřující výchozí limity (pro zajímavost); **Odkaz** na důkazy platnosti jednotlivých limit.

Zadání

1. Vypočtěte limity:

   a) $\lim_{x\to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x-a}$

   b) $\lim_{x\to 0} \frac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x}$

   c) $\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x}$

   d) $\lim_{x\to 0} \frac{1-\cos x \cos 2x \cos 3x}{1-\cos x}$

   e) $\lim_{x\to a} \frac{\mathrm{tg} x - \mathrm{tg} a}{x-a}$

   f) $\lim_{x\to \frac{\pi}{2}} \frac{e^{\sin\cos x}-1}{\mathrm{cotg}\, x}$

2. Vypočtěte limity:

   a) $\lim_{x\to 0} \frac{\ln(\cos ax)}{\ln(\cos bx)}$

   b) $\lim_{x\to a} \frac{\log_a x - \log_x a}{x-a}$

   c) $\lim_{x\to 1} (1-x)\log_x 2$

3. Vypočtěte limity:

   a) $\lim_{x\to \infty} \left(1-\frac{1}{x}\right)^x$

   b) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\mathrm{tg} x}$

   c) $\lim_{x\to 0} \left(\frac{1+\mathrm{tg} x}{1+\sin x}\right)^{\frac{1}{\sin x}}$

   d) $\lim_{x\to \infty} \left(\frac{x+a}{x-a}\right)^x$

4. Vypočtěte limity:

   a) $\lim_{x\to \infty} \frac{e^{1+\ln x}}{\ln(e^{3x}+e^{-3x})}$

   b) $\lim_{x\to 4} \frac{x^2+7x-44}{x^2-6x+8}$

   c) $\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2}$

   d) $\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x^2})}{\sin x}$

   e) $\lim_{x\to \infty} \left(\frac{1-2x}{5-2x}\right)^x$

   f) $\lim_{x\to 10} \frac{\log x - \log 10}{\log \frac{10}{x}}$

   g) $\lim_{x\to 0_+} \left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$

   h) $\lim_{x\to 1} \left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$

   i) $\lim_{x\to -\infty} (\sqrt{9x^2+(2x-1)} - \sqrt{9x^2-(2x-1)})$

   j) $\lim_{x\to \infty} \frac{1}{\sqrt{9x+2x^2}-x\sqrt{2}}$

   k) $\lim_{x\to \infty} \frac{\log(x^2-x+1)}{\log(x^{10}+x+1)}$

   l) $\lim_{x\to -1} \left(\frac{\arcsin 2^x}{\pi+\mathrm{arccotg} x}\right)$

5. Vypočtěte limity:

a) $\lim\limits_{x\to -1} \frac{x^3+1}{\sin(x+1)}$

b) $\lim\limits_{x\to 0} x \cdot \frac{e^{\sin x}-1}{1-\cos x}$

c) $\lim\limits_{x\to 0} \frac{\cos x-\cos 3x}{x^2}$

d) $\lim\limits_{x\to \frac{\pi}{6}} \frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$.

e) $\lim\limits_{x\to 0} \frac{(\sin 3x-\sin x)(e^{4x^2-3x+1}-e)}{\ln((x+1)(2x+1)(3x+1))(\sqrt{x^3+6x+4}-2)}$

f) $\lim\limits_{x\to 0} \frac{(\cos 3x-\cos x)(2^2-2^{2(x+1)})}{\ln((x+1)(3x+1)(5x+1))(\sqrt{x^3+x^2+9}-3)}$.

Řešení

1. Pro většinu úloh z tohoto cvičení využijeme limity $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{e^x-1}{x}=1$, $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$:

a)

$$\begin{aligned}
\lim_{x\to a}\frac{\frac{1}{\sin x}-\frac{1}{\sin a}}{x-a} &= \lim_{x\to a}\frac{\sin a-\sin x}{\sin x\sin a\,(x-a)} = \lim_{x\to a}\frac{\sin a-\sin(x-a+a)}{\sin x\sin a\,(x-a)}\\
&= \frac{1}{\sin a}\cdot\lim_{x\to a}\frac{1}{\sin x}\cdot\lim_{x\to a}\frac{\sin a-(\sin(x-a)\cos a+\cos(x-a)\sin a)}{(x-a)}\\
&= \frac{1}{\sin a}\cdot\frac{1}{\sin a}\cdot\lim_{x\to a}\frac{\sin a-\cos(x-a)\sin a-\sin(x-a)\cos a}{(x-a)}\\
&= \frac{1}{\sin^2 a}\cdot\lim_{x\to a}\frac{\sin a(1-\cos(x-a))-\sin(x-a)\cos a}{(x-a)}\\
&= \frac{1}{\sin^2 a}\cdot\lim_{x\to a}\frac{\sin a(1-\cos(x-a))}{(x-a)}-\frac{1}{\sin^2 a}\cdot\lim_{x\to a}\frac{\sin(x-a)\cos a}{(x-a)}\\
&= \frac{1}{\sin a}\cdot\lim_{x\to a}\frac{(1-\cos^2(x-a))}{(x-a)(1+\cos(x-a))}-\frac{1}{\sin^2 a}\cdot 1\cdot\cos a\\
&= \frac{1}{\sin a}\cdot\lim_{x\to a}\frac{(x-a)\sin^2(x-a)}{(x-a)^2(1+\cos(x-a))}-\frac{\cos a}{\sin^2 a}\\
&= \frac{1}{\sin a}\cdot\lim_{x\to a}\frac{x-a}{1+\cos(x-a)}\cdot\left(\frac{\sin(x-a)}{(x-a)}\right)^2-\frac{\cos a}{\sin^2 a}\\
&= \frac{1}{\sin a}\cdot\frac{0}{1+\cos(0)}\cdot 1^2-\frac{\cos a}{\sin^2 a} = -\frac{\cos a}{\sin^2 a}
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x} &= \lim_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}\\
&\cdot\frac{\sqrt{\cos x}^5+\sqrt{\cos x}^4\sqrt[3]{\cos x}+\sqrt{\cos x}^3\sqrt[3]{\cos x}^2+\sqrt{\cos x}^2\sqrt[3]{\cos x}^3+\sqrt{\cos x}\sqrt[3]{\cos x}^4+\sqrt[3]{\cos x}^5}{\sqrt{\cos x}^5+\sqrt{\cos x}^4\sqrt[3]{\cos x}+\sqrt{\cos x}^3\sqrt[3]{\cos x}^2+\sqrt{\cos x}^2\sqrt[3]{\cos x}^3+\sqrt{\cos x}\sqrt[3]{\cos x}^4+\sqrt[3]{\cos x}^5}\\
&= \lim_{x\to 0}\frac{\cos^3 x-\cos^2 x}{\sin^2 x}\\
&\cdot\lim_{x\to 0}\frac{1}{\sqrt{\cos x}^5+\sqrt{\cos x}^4\sqrt[3]{\cos x}+\sqrt{\cos x}^3\sqrt[3]{\cos x}^2+\sqrt{\cos x}^2\sqrt[3]{\cos x}^3+\sqrt{\cos x}\sqrt[3]{\cos x}^4+\sqrt[3]{\cos x}^5}\\
&= -\lim_{x\to 0}\frac{\cos^2 x(1-\cos x)}{\sin^2 x}\cdot\frac{1}{6} = -\frac{1}{6}\lim_{x\to 0}\frac{\cos^2 x(1-\cos x)}{\sin^2 x}\cdot\frac{x^2}{x^2}\\
&= -\frac{1}{6}\lim_{x\to 0}\cos^2 x\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}\frac{x^2}{\sin^2 x} = -\frac{1}{6}\cdot 1^2\cdot\frac{1}{2}\cdot\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = -\frac{1}{12}\cdot 1^2 = -\frac{1}{12}
\end{aligned}$$

c)

$$\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x}\cdot\frac{x^2}{x^2} = \lim_{x\to 0}\sqrt{\frac{1-\cos x^2}{x^4}}\cdot\lim_{x\to 0}\frac{x^2}{1-\cos x} = \sqrt{\frac{1}{2}}\cdot\frac{2}{1} = \sqrt{2}$$

d)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x} &= \lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}\frac{x^2}{x^2} = \lim_{x\to 0}\frac{x^2}{1-\cos x}\cdot\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{x^2}\\
&= \frac{2}{1}\cdot\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{x^2}\\
&= 2\cdot\lim_{x\to 0}\frac{1-\cos x(\cos^2 x-\sin^2 x)(\cos x\cos 2x-\sin x\sin 2x)}{x^2}\\
&= 2\cdot\lim_{x\to 0}\frac{1-\cos x(\cos^2 x-\sin^2 x)(\cos x(\cos^2 x-\sin^2 x)-\sin x\cdot 2\sin x\cos x)}{x^2}\\
&= 2\cdot\lim_{x\to 0}\frac{1-\cos x(\cos^2 x-\sin^2 x)(\cos^3 x-3\sin^2 x\cos x)}{x^2}\\
&= 2\cdot\lim_{x\to 0}\frac{1-\cos^6 x+4\sin^2 x\cos^4 x-\sin^4 x\cos^2 x}{x^2}\\
&= 2\left(\lim_{x\to 0}\frac{1-\cos^6 x}{x^2}+\lim_{x\to 0}\frac{4\sin^2 x\cos^4 x}{x^2}-\lim_{x\to 0}\frac{\sin^4 x\cos^2 x}{x^2}\right)\\
&= 2\left(3+4-0\right)=2\cdot 7=14
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos^6 x}{x^2} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x+...+\cos^5 x)}{x^2}\\
&= \lim_{x\to 0}\frac{(1-\cos x)}{x^2}\cdot(1+\cos x+\cos^2 x+...+\cos^5 x)=\frac{1}{2}\cdot 6=3
\end{aligned}$$

$$\lim_{x\to 0}\frac{4\sin^2 x\cos^4 x}{x^2} = 4\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\cos^4 x = 4\cot 1\cdot 1 = 4$$

$$\lim_{x\to 0}\frac{\sin^4 x\cos^2 x}{x^2} = \lim_{x\to 0}\frac{\sin^4 x}{x^4}\cdot\frac{x^2\cos^2 x}{1} = 1\cdot 0 = 0$$

e)

$$\lim_{x\to a}\frac{\operatorname{tg} x-\operatorname{tg} a}{x-a} = \lim_{x\to a}\frac{\sin(x-a)}{\cos x\cos a(x-a)} = \lim_{x\to a}\frac{\sin(x-a)}{(x-a)}\cdot\lim_{x\to a}\frac{1}{\cos x\cos a} = 1\cdot\frac{1}{\cos^2 a} = \frac{1}{\cos^2 a}$$

(Krom užité identity $\operatorname{tg}\alpha\pm\operatorname{tg}\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$ lze postupovat i poněkud méně elegantně skrze vztah $\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$. Lze si také uvědomit, že hledaná limita je z definice derivace funkce $\operatorname{tg} x$ v bodě a a tedy je rovna $\frac{1}{\cos^2 a}$.)

f)

$$\lim_{x\to\frac{\pi}{2}}\frac{e^{\sin\cos x}-1}{\operatorname{cotg} x} = \lim_{x\to\frac{\pi}{2}}\frac{e^{\sin\cos x}-1}{\operatorname{cotg} x}\cdot\frac{\sin\cos x}{\sin\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{e^{\sin\cos x}-1}{\sin\cos x}\cdot\lim_{x\to\frac{\pi}{2}}\frac{\sin\cos x}{\frac{\cos x}{\sin x}} = 1\cdot\lim_{x\to\frac{\pi}{2}}\frac{\sin\cos x}{\cos x}\sin x = 1\cdot 1\cdot 1 = 1$$

2. Pro většinu úloh z tohoto cvičení využijeme limity $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{\ln(x+1)}{x}=1$, $\lim_{x\to 1}\frac{\ln x}{x-1}=1$:

a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)} &= \lim_{x\to 0}\frac{\ln(\cos ax+1-1)}{\ln(\cos bx+1-1)} = \lim_{x\to 0}\frac{\ln((\cos ax-1)+1)}{\ln((\cos bx-1)+1)}\cdot\frac{\cos ax-1}{\cos ax-1}\cdot\frac{\cos bx-1}{\cos bx-1}\\
&= \lim_{x\to 0}\frac{\ln((\cos ax-1)+1)}{\cos ax-1}\cdot\lim_{x\to 0}\frac{\cos bx-1}{\ln((\cos bx-1)+1)}\cdot\lim_{x\to 0}\frac{\cos ax-1}{\cos bx-1}\\
&= 1\cdot 1\cdot\lim_{x\to 0}\frac{\cos ax-1}{\cos bx-1}\cdot\frac{(ax)^2(bx)^2}{(ax^2)(bx)^2} = \lim_{x\to 0}\frac{\cos ax-1}{(ax)^2}\cdot\lim_{x\to 0}\frac{(bx)^2}{\cos bx-1}\cdot\lim_{x\to 0}\frac{a^2x^2}{b^2x^2}\\
&= -\frac{1}{2}\cdot\left(-\frac{2}{1}\right)\cdot\frac{a^2}{b^2} = \frac{a^2}{b^2}
\end{aligned}$$

b)

$$\lim_{x\to a}\frac{\log_a x-\log_x a}{x-a}=\lim_{x\to a}\frac{\frac{\ln x}{\ln a}-\frac{\ln a}{\ln x}}{x-a}=\lim_{x\to a}\frac{\frac{\ln^2 x-\ln^2 a}{\ln a\ln x}}{x-a}=\lim_{x\to a}\frac{1}{\ln a\ln x}\cdot\lim_{x\to a}\frac{\ln^2 x-\ln^2 a}{x-a}$$
$$=\frac{1}{\ln^2 a}\lim_{x\to a}\frac{\ln x-\ln a}{x-a}\cdot\lim_{x\to a}(\ln x+\ln a)=\frac{2\ln a}{\ln^2 a}\lim_{x\to a}\frac{\ln\left(\frac{x}{a}\right)}{a\left(\frac{x}{a}-1\right)}=\frac{2}{a\ln a}\lim_{x\to a}\frac{\ln\left(\frac{x}{a}\right)}{\frac{x}{a}-1}$$
$$=\frac{2}{a\ln a}\cdot 1=\frac{2}{a\ln a}$$

c)

$$\lim_{x\to 1}(1-x)\log_x 2=-\lim_{x\to 1}(x-1)\cdot\frac{\ln 2}{\ln x}=-\ln 2\cdot\lim_{x\to 1}\frac{x-1}{\ln x}=-\ln 2\cdot 1=-\ln 2$$

3. Pro většinu úloh z tohoto cvičení využijeme limity $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\mathrm{e}$, $\lim_{x\to 0}\frac{\ln(x+1)}{x}=1$, $\lim_{x\to 1}\frac{\ln x}{x-1}=1$ a vztahu $a^b=\mathrm{e}^{b\ln a}$:

a)

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(1+\frac{1}{-x}\right)^{-x\cdot(-1)}=\lim_{x\to\infty}\left(\left(1+\frac{1}{-x}\right)^{-x}\right)^{-1}=\mathrm{e}^{-1}=\frac{1}{\mathrm{e}}$$

b)

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\mathrm{tg}x}=\lim_{x\to\frac{\pi}{2}}e^{\mathrm{tg}x\cdot\ln(\sin x)}=\lim_{x\to\frac{\pi}{2}}e^{\mathrm{tg}x\cdot\frac{\ln(\sin x)}{\sin x-1}\cdot(\sin x-1)}=e^{0}=1$$

$$\lim_{x\to\frac{\pi}{2}}\mathrm{tg}x\cdot\frac{\ln(\sin x)}{\sin x-1}\cdot(\sin x-1)=\lim_{x\to\frac{\pi}{2}}\sin x\cdot\frac{\ln(\sin x)}{\sin x-1}\cdot\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}1\cdot 1\cdot\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}$$
$$=\lim_{y\to 0}\frac{\sin(y+\frac{\pi}{2})-1}{\cos(y+\frac{\pi}{2})}=\lim_{y\to 0}\frac{\cos y-1}{\cos y\cos\frac{\pi}{2}-\sin y\sin\frac{\pi}{2}}$$
$$=\lim_{y\to 0}\frac{\cos y-1}{\cos y\cdot 0-\sin y\cdot 1}=\lim_{y\to 0}\frac{\cos y-1}{-\sin y}=\lim_{y\to 0}\frac{\cos y-1}{y^2}\cdot\frac{y}{-\sin y}\cdot y$$
$$=\frac{-1}{2}\cdot(-1)\cdot 0=0$$

c)

$$\lim_{x\to 0}\left(\frac{1+\mathrm{tg}x}{1+\sin x}\right)^{\frac{1}{\sin x}}=\lim_{x\to 0}\mathrm{e}^{\left(\frac{1}{\sin x}\right)\ln\left(\frac{1+\mathrm{tg}x}{1+\sin x}\right)}=\ldots=\mathrm{e}^{0}=1$$

$$\lim_{x\to 0}\frac{1}{\sin x}\ln\left(\frac{1+\mathrm{tg}x}{1+\sin x}\right)=\lim_{x\to 0}\frac{1}{\sin x}\cdot\frac{\ln\left(\frac{1+\mathrm{tg}x}{1+\sin x}\right)}{\frac{1+\mathrm{tg}x}{1+\sin x}-1}\cdot\left(\frac{1+\mathrm{tg}x}{1+\sin x}-1\right)$$
$$=\lim_{x\to 0}\frac{1}{\sin x}\cdot 1\cdot\frac{1+\mathrm{tg}x-1-\sin x}{1+\sin x}=\lim_{x\to 0}\frac{1}{\sin x}\cdot\frac{\frac{\sin x-\sin x\cos x}{\cos x}}{1+\sin x}$$
$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x(1+\sin x)}=0$$

d)

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x=\lim_{x\to\infty}e^{x\ln\left(\frac{x+a}{x-a}\right)}=\lim_{x\to\infty}e^{x\cdot\frac{\ln\left(\frac{x+a}{x-a}\right)}{\frac{x+a}{x-a}-1}\cdot\left(\frac{x+a}{x-a}-1\right)}=\lim_{x\to\infty}e^{x\cdot 1\cdot\frac{2a}{x-a}}=e^{2a}$$

4. a)

$$\lim_{x\to\infty} \frac{e^{1+\ln x}}{\ln(e^{3x}+e^{-3x})} = \lim_{x\to\infty} \frac{e\cdot e^{\ln x}}{\ln(e^{3x}(1+e^{-6x}))} = e\lim_{x\to\infty} \frac{x}{\ln(e^{3x})+\ln(1+e^{-6x})} = e\lim_{x\to\infty} \frac{x}{3x+\ln(1+e^{-6x})}$$
$$= e\lim_{x\to\infty} \frac{x}{x\left(3+\frac{\ln(1+e^{-6x})}{x}\right)} = \frac{e}{3}$$

b)

$$\lim_{x\to 4} \frac{x^2+7x-44}{x^2-6x+8} = \lim_{x\to 4} \frac{(x-4)(x+11)}{(x-4)(x-2)} = \lim_{x\to 4} \frac{(x+11)}{(x-2)} = \frac{15}{2}$$

c)

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{x+2-4} = 4$$

d)

$$\lim_{x\to 0} \frac{x^2\sin(\frac{1}{x^2})}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x}\cdot x\cdot\sin(\frac{1}{x^2}) = \lim_{x\to 0} x\cdot\sin(\frac{1}{x^2}) = 0$$

e)

$$\lim_{x\to\infty} \left(\frac{1-2x}{5-2x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(\frac{1-2x}{5-2x}\right)} = \lim_{x\to\infty} e^{x\cdot\frac{\ln\left(\frac{1-2x}{5-2x}\right)}{\frac{1-2x}{5-2x}-1}\cdot\left(\frac{1-2x}{5-2x}-1\right)} = \lim_{x\to\infty} e^{x\cdot 1\cdot\frac{1-2x-5+2x}{5-2x}} = e^2$$

f)

$$\lim_{x\to 10} \frac{\log x-\log 10}{\log\frac{10}{x}} = \lim_{x\to 10} \frac{\log\frac{x}{10}}{\log\frac{10}{x}} = \lim_{x\to 10} \frac{\log\left(\frac{10}{x}\right)^{-1}}{\log\frac{10}{x}} = \lim_{x\to 10} \frac{-1\cdot\log\frac{10}{x}}{\log\frac{10}{x}} = -1$$

g)

$$\lim_{x\to 0_+} \left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \lim_{x\to 0_+} e^{\frac{1-\sqrt{x}}{1-x}\cdot\ln\left(\frac{1+x}{2+x}\right)} = e^{1\cdot\ln\left(\frac{1}{2}\right)} = \frac{1}{2}$$

h)

$$\lim_{x\to 1} \left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \lim_{x\to 1} e^{\frac{1-\sqrt{x}}{1-x}\cdot\ln\left(\frac{1+x}{2+x}\right)} = \lim_{x\to 1} e^{\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\cdot\ln\left(\frac{1+x}{2+x}\right)} \lim_{x\to 1} e^{\frac{1}{1+\sqrt{x}}\cdot\ln\left(\frac{1+x}{2+x}\right)} = e^{\frac{1}{2}\cdot\ln\frac{2}{3}} = \sqrt{\frac{2}{3}}$$

i)

$$\lim_{x\to-\infty}(\sqrt{9x^2+(2x-1)}-\sqrt{9x^2-(2x-1)}) = \lim_{x\to-\infty}(\sqrt{9x^2+(2x-1)}-\sqrt{9x^2-(2x-1)})$$
$$\cdot\frac{\sqrt{9x^2+(2x-1)}+\sqrt{9x^2-(2x-1)}}{\sqrt{9x^2+(2x-1)}+\sqrt{9x^2-(2x-1)}}$$
$$= \lim_{x\to-\infty} \frac{9x^2+(2x-1)-(9x^2-(2x-1))}{\sqrt{9x^2+(2x-1)}+\sqrt{9x^2-(2x-1)}}$$
$$= \lim_{x\to-\infty} \frac{4x}{\sqrt{x^2}(\sqrt{9+\frac{2x-1}{x^2}}+\sqrt{9-\frac{2x-1}{x^2}})}$$
$$= \lim_{x\to-\infty} \frac{4x}{|x|\cdot(\sqrt{9+\frac{2x-1}{x^2}}+\sqrt{9-\frac{2x-1}{x^2}})}$$
$$= \lim_{x\to-\infty} \frac{4x}{-x\cdot(\sqrt{9+\frac{2x-1}{x^2}}+\sqrt{9-\frac{2x-1}{x^2}})}$$
$$= \lim_{x\to-\infty} \frac{-4}{\sqrt{9+\frac{2x-1}{x^2}}+\sqrt{9-\frac{2x-1}{x^2}}} = \frac{-4}{6} = \frac{-2}{3}$$

j)

$$\lim_{x\to\infty}\frac{1}{\sqrt{9x+2x^2}-x\sqrt{2}}=\lim_{x\to\infty}\frac{1}{\sqrt{9x+2x^2}-x\sqrt{2}}\cdot\frac{\sqrt{9x+2x^2}+x\sqrt{2}}{\sqrt{9x+2x^2}+x\sqrt{2}}=\lim_{x\to\infty}\frac{\sqrt{9x+2x^2}+x\sqrt{2}}{9x+2x^2-2x^2}$$
$$\lim_{x\to\infty}\frac{x(\sqrt{\frac{9}{x}+2}+\sqrt{2})}{9x}=\frac{2\sqrt{2}}{9}$$

k)

$$\lim_{x\to\infty}\frac{\log(x^2-x+1)}{\log(x^{10}+x+1)}=\lim_{x\to\infty}\frac{\log(x^2(1-\frac{1}{x}+\frac{1}{x^2}))}{\log(x^{10}(1+\frac{1}{x^9}+\frac{1}{x^{10}}))}=\lim_{x\to\infty}\frac{\log(x^2)+\log(1-\frac{1}{x}+\frac{1}{x^2})}{\log(x^{10})+\log(1+\frac{1}{x^9}+\frac{1}{x^{10}})}$$
$$=\lim_{x\to\infty}\frac{2\log x+\log(1-\frac{1}{x}+\frac{1}{x^2})}{10\log x+\log(1+\frac{1}{x^9}+\frac{1}{x^{10}})}=\lim_{x\to\infty}\frac{2+\frac{\log(1-\frac{1}{x}+\frac{1}{x^2})}{\log x}}{10+\frac{\log(1+\frac{1}{x^9}+\frac{1}{x^{10}})}{\log x}}=\frac{1}{5}$$

l)

$$\lim_{x\to-1}\left(\frac{\operatorname{arcsin}2^x}{\pi+\operatorname{arccotg}x}\right)=\left(\frac{\operatorname{arcsin}\frac{1}{2}}{\pi+\operatorname{arccotg}(-1)}\right)=\frac{\frac{\pi}{6}}{\pi+\frac{3}{4}\pi}=\frac{2}{21}$$

5. a)

$$\lim_{x\to-1}\frac{x^3+1}{\sin(x+1)}=\lim_{y\to0}\frac{(y-1)^3+1}{\sin(y)}=\lim_{y\to0}\frac{y^3-3y^2+3y}{\sin(y)}=\lim_{y\to0}\frac{y}{\sin(y)}\cdot(y^2-3y+3)=1\cdot3=3$$

b)

$$\lim_{x\to0}x\cdot\frac{e^{\sin x}-1}{1-\cos x}=\lim_{x\to0}\frac{e^{\sin x}-1}{\sin x}\cdot\frac{\sin x}{x}\cdot\frac{x^2}{1-\cos x}=1\cdot1\cdot2=2$$

c)

$$\lim_{x\to0}\frac{\cos x-\cos 3x}{x^2}=\lim_{x\to0}\frac{\cos x-(\cos x\cos 2x-\sin x\sin 2x)}{x^2}=\lim_{x\to0}\frac{\cos x-(\cos x(\cos^2 x-\sin^2 x)-\sin x\cdot 2\sin x\cos x)}{x^2}$$
$$=\lim_{x\to0}\frac{\cos x-\cos^3 x+3\cos x\sin^2 x}{x^2}=\lim_{x\to0}\cos x\cdot\lim_{x\to0}\left(\frac{1-\cos^2 x}{x^2}+\frac{3\sin^2 x}{x^2}\right)$$
$$=1\cdot\lim_{x\to0}\left(\frac{(1-\cos x)(1+\cos x)}{x^2}+\frac{3\sin^2 x}{x^2}\right)=\lim_{x\to0}\frac{1-\cos x}{x^2}\cdot(1+\cos x)+\lim_{x\to0}\frac{3\sin^2 x}{x^2}$$
$$=\frac{1}{2}\cdot2+3=4$$

d)

$$\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}=\lim_{x\to\frac{\pi}{6}}\frac{(\sin x-\frac{1}{2})(2\sin x+2)}{(\sin x-\frac{1}{2})(2\sin x-2)}=\lim_{x\to\frac{\pi}{6}}\frac{2\sin x+2}{2\sin x-2}=\frac{1+2}{1-2}=-3$$

e)

$$\lim_{x\to0}\frac{(\sin 3x-\sin x)(e^{4x^2-3x+1}-e)}{\ln((x+1)(2x+1)(3x+1))(\sqrt{x^3+6x+4}-2)}=\lim_{x\to0}\frac{(\sin 3x-\sin x)\cdot e\cdot\frac{(e^{4x^2-3x}-1)}{x^2-3x}\cdot(4x^2-3x)}{\ln((x+1)(2x+1)(3x+1))(\sqrt{x^3+6x+4}-2)}$$
$$=\lim_{x\to0}\frac{e\cdot1\cdot x(\sin 3x-\sin x)(4x-3)}{\ln((x+1)(2x+1)(3x+1))(\sqrt{x^3+6x+4}-2)}$$
$$=\lim_{x\to0}\frac{-3\cdot e\cdot1\cdot x(\sin 3x-\sin x)}{\frac{\ln((x+1)(2x+1)(3x+1))}{(x+1)(2x+1)(3x+1)-1}((x+1)(2x+1)(3x+1)-1)(\sqrt{x^3+6x+4}-2)}$$
$$=\lim_{x\to0}\frac{-3e\cdot x(\sin 3x-\sin x)}{1\cdot(6x+o(x))(\sqrt{x^3+6x+4}-2)}\cdot\frac{\sqrt{x^3+6x+4}+2}{\sqrt{x^3+6x+4}+2}$$
$$=\lim_{x\to0}\frac{-3e\cdot x(\sin 3x-\sin x)}{(6x+o(x))(x^3+6x+4-4)}\cdot\frac{\sqrt{x^3+6x+4}+2}{1}=\lim_{x\to0}\frac{-3e\cdot x(\frac{\sin 3x}{3x}\cdot3x-\frac{\sin x}{x}\cdot x)}{(6x+o(x))(6x+o(x))}\cdot4$$
$$=\lim_{x\to0}\frac{-12e\cdot x^2(\frac{\sin 3x}{3x}\cdot3-\frac{\sin x}{x})}{36x^2+o(x^2)}=\lim_{x\to0}\frac{-12e\cdot x^2\cdot(-2)}{36x^2+o(x^2)}=\lim_{x\to0}\frac{-24e}{36+\frac{o(x^2)}{x^2}}=\frac{-2e}{3}$$

f)

$$\lim_{x\to 0}\frac{(\cos 3x-\cos x)(2^2-2^{2(x+1)})}{\ln((x+1)(3x+1)(5x+1))(\sqrt{x^3+x^2+9}-3)}=\lim_{x\to 0}\frac{-2^2(\cos 3x-\cos x)(2^{2(x+1)-2}-1)}{\ln((x+1)(3x+1)(5x+1))(\sqrt{x^3+x^2+9}-3)}$$

$$=\lim_{x\to 0}\frac{-2^2(\cos 3x-\cos x)(e^{2x\ln 2}-1)}{\frac{\ln((x+1)(3x+1)(5x+1))}{(x+1)(3x+1)(5x+1)+1}\cdot(x+1)(3x+1)(5x+1)-1)(\sqrt{x^3+x^2+9}-3)}$$

$$=\lim_{x\to 0}\frac{-4(\cos 3x-\cos x)\cdot\frac{e^{2x\ln 2}-1}{2x\ln 2}\cdot(2x\ln 2)}{1\cdot(9x+o(x))(\sqrt{x^3+x^2+9}-3)}\cdot\frac{\sqrt{x^3+x^2+9}+3}{\sqrt{x^3+x^2+9}+3}$$

$$=\lim_{x\to 0}\frac{-4(\cos 3x-\cos x)\cdot 1\cdot 2x\ln 2}{(9x+o(x))(x^3+x^2+9-9)}\cdot 6=\lim_{x\to 0}\frac{-48\ln 2(\cos 3x-\cos x)\cdot x}{(9x+o(x))(x^2+o(x^2))}$$

$$\lim_{x\to 0}\frac{-48\ln 2(\cos 3x-\cos x)\cdot x}{9x^3+o(x^3)}=\lim_{x\to 0}\frac{48\ln 2\cdot\frac{\cos x-\cos 3x}{x^2}\cdot x^3}{9x^3+o(x^3)}=\lim_{x\to 0}\frac{48\ln 2}{9+\frac{o(x^3)}{x^3}}\cdot\frac{\cos x-\cos 3x}{x^2}$$

$$=\frac{16\ln 2}{3}\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=\frac{16\ln 2}{3}\cdot 4=\frac{64\ln 2}{3}$$

Postup výpočtu limity $\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}$ je uveden ve cvičení 5.c).