Příklady jsou převzaty ze stránky doktorky Pavlíkové (http//:www.karlin.mff.cuni.cz/~pavlikov/).

Zadání

1. Ukažte, že funkce $F_1(x)$ a $F_2(x)$ jsou primitivní funkce k téže funkci, a určete konstantu, o kterou se liší.
   a) $F_1(x) = \ln\sqrt{x-2} + 3,\ F_2(x) = \ln\sqrt{2x-4}$
   b) $F_1(x) = \cos 2x,\ F_2(x) = 6\cos^2 x + 4\sin^2 x$
2. Ukažte, že funkce $F(x) = x(\text{arctg}x + \text{arctg}\frac{1}{x}) + \pi$ je primitivní funkce k funkci $f(x) = \frac{\pi}{2}$ na intervalu $(0, \infty)$.
3. Určete primitivní funkce.
   a) $\int(x^3 + x^2 - 7)dx$
   b) $\int \frac{x^2}{\sqrt{x}}dx$
   c) $\int(1 + 2x)^3 dx$
   d) $\int \frac{x^2+2x}{x-1}dx$
   e) $\int \frac{x^3+1}{x+1}dx$
   f) $\int(\sin x - 2\cos x)dx$
   g) $\int(\cos 3x + 3x + 1)dx$
4. Určete primitivní funkce.
   a) $\int \sqrt{x\sqrt{x\sqrt{x}}}dx$
   b) $\int \sin x \cos x dx$
   c) $\int e^x \cdot 5^{x-1} dx$
   d) $\int \frac{x}{|x|}dx$
   e) $\int x^5 dy$

5. Určete primitivní funkce (metodou substituce).

   a) $\int \frac{2}{\sqrt{1-4x^2}} dx$
   b) $\int \frac{1}{x^2+9} dx$
   c) $\int \frac{1}{4x^2+3} dx$
   d) $\int x\sqrt{x^2-4} dx$
   e) $\int \frac{e^x}{1+e^x} dx$
   f) $\int \frac{1}{1+e^{-x}} dx$
   g) $\int \frac{3^x}{1+9^x} dx$
   h) $\int \frac{1}{x^2+4x+5} dx$

6. Určete primitivní funkce.

   a) $\int \mathrm{tg}^2 x dx$
   b) $\int \frac{\sin 2x}{5\cos x} dx$
   c) $\int \sin^2 \frac{x}{2}$
   d) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$
   e) $\int \mathrm{cotg} x dx$
   f) $\int \sin^3 x dx$
   g) $\int \sqrt{1-x^2} dx$
   h) $\int x^2\sqrt{x^3+5} dx$
   i) $\int x^2 \sin x dx$
   j) $\int \cos^3 x \sin x dx$
   k) $\int \frac{\ln x}{x} dx$
   l) $\int \mathrm{arcsin} x dx$
   m) $\int \sin x \cdot \ln \mathrm{tg} x dx$
   n) $\int \ln(x+\sqrt{1+x^2}) dx$
   o) $\int \frac{\ln \sin x}{\sin^2 x} dx$
   p) $\int \sin x \cdot \sin 2x \cdot \sin 3x dx$

7. Integrace racionálních funkcí:

   a) $\int (\frac{5}{3x-2} + \frac{1}{(1-4x)^2}) dx$
   b) $\int \frac{2x-3}{x^2-3x+5} dx$
   c) $\int \frac{1}{x^2-2x+5} dx$
   d) $\int \frac{x-4}{2x^2-x+2} dx$
   e) $\int \frac{1}{x^2+2x} dx$
   f) $\int \frac{x^4}{x^4+5x^2+4} dx$
   g) $\int (\frac{x}{x^2-3x+2})^2 dx$
   h) $\int \frac{x^3}{(x-1)(x-2)^2} dx.$

8. Určete primitivní funkce.

   a) $\int \frac{1}{1-\cos x} dx$
   b) $\int \cos^3 x dx$
   c) $\int \frac{1}{\cos^4 x} dx$

d) $\int \mathrm{tg}^2 x - \mathrm{tg} x dx$

e) $\int \frac{1}{\sin^2 x - \sin x \cos x - 2\cos^2 x} dx$

f) $\int \frac{2\sin x - 3\cos x}{\sin x + 5\cos x} dx$

9. Určete primitivní funkce.

a) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}} dx$

b) $\int \sqrt{\frac{a+x}{a-x}} dx$

c) $\int \frac{1}{\sqrt{1-4x-x^2}} dx$

d) $\int \frac{1}{\sqrt{1+x+x^2}} dx$

e) $\int \frac{1}{\sqrt{9-4x^2}} dx$

f) $\int \frac{1}{x\sqrt{1+x^2}} dx$

10. Určete primitivní funkce.

a) $\int \frac{\mathrm{arctg}\sqrt{x}}{\sqrt{x}} \frac{1}{1+x} dx$

b) $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

c) $\int \frac{e^x}{e^{2x}-1} dx$

d) $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}} dx$

e) $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}} dx$

f) $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx$

g) $\int \frac{1}{\sin^2(2x+\frac{\pi}{4})} dx$

h) $\int \frac{x^3}{x^8-2} dx$

i) $\int \frac{1}{(1+x)\sqrt{x}} dx$

j) $\int \frac{1}{\sqrt{1+e^{2x}}} dx$

k) $\int \frac{1}{x \ln x \ln(\ln x)} dx$

l) $\int \frac{1}{x\sqrt{x^2-1}} dx$

m) $\frac{\sin x}{\sqrt{\cos^3 x}} dx$

n) $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$

Řešení

1. a)

$$F_2(x) = \ln\sqrt{2x-4} = \ln\left(\sqrt{2}\cdot\sqrt{x-2}\right) = \ln\sqrt{2} + \ln\sqrt{x-2} = (\ln\sqrt{2}-3) + \ln\sqrt{x-2} + 3 = (\ln\sqrt{2}-3) + F_1(x).$$

   b)

$$F_1(x) = \cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - \sin^2(x) + 5 - 5 = 6\cos^2(x) + 4\sin^2(x) - 5 = F_2(x) + 5.$$

2. $F(x) = x(\operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right)) + \pi\ f(x) = \frac{\pi}{x}$.

$$F'(x) = \left(\operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right)\right) + x\left(\frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2}\right) = \operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right) + x\left(\frac{1}{1+x^2} + \frac{x^2}{1+x^2}\cdot\frac{-1}{x^2}\right)$$
$$= \operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right) \overset{(*)}{=} \frac{\pi}{2}.$$

Poslední rovnost (označená $(*)$) vychází z následujícího výpočtu.

$$\left(\operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right)\right)' = \left(\frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2}\right) = 0,$$

tedy $\operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right)$ je konstantí funkce. Dále stačí dosadit například $x = 1$ ($\operatorname{arctg}(1) = \frac{\pi}{4}$).

3. a) $\int(x^3+x^2-7)dx = \frac{x^4}{4} + \frac{x^3}{3} - 7x + C$

   b) $\int \frac{x^2}{\sqrt{x}}dx = \int x^{\frac{3}{2}}dx = \frac{2x^{\frac{5}{2}}}{5} + C$

   c) $\int(1+2x)dx = \left|\begin{array}{c} z = 1+2x \\ dz = 2dx \end{array}\right| = \frac{1}{2}\int z dz = \frac{z^2}{4} + C = \frac{(1+2x)^2}{4} + C$

   d) $\int \frac{x^2+2x}{x-1}dx = \int\left(x+3+\frac{3}{x-1}\right)dx = \frac{x^2}{2} + 3x + 3\ln|x-1| + C$

   e) $\int \frac{x^3+1}{x+1}dx = \int\left(x^2-x+1\right)dx = \frac{x^3}{3} + \frac{-x^2}{2} + x + C$

   f) $\int(\sin x - 2\cos x)dx = -\cos x - 2\sin x + C$

   g) $\int(\cos 3x + 3x + 1)dx = \frac{\sin 3x}{3} + \frac{3x^2}{2} + x + C$

4. a) $\int\sqrt{x\sqrt{x\sqrt{x}}}dx = \int x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}dx = \int x^{\frac{7}{8}}dx = \frac{8x^{\frac{15}{8}}}{15} + C$

   b) I. $\int \sin x\cos x dx = \left|\begin{array}{c} z = \sin x \\ dz = \cos x dx \end{array}\right| = \int z dz = \frac{z^2}{2} + C = \frac{\sin^2(x)}{2} + C$

   II. $\int \sin x\cos x dx = \frac{1}{2}\int\sin(2x)dx = \frac{-1}{2}\cdot\frac{\cos(2x)}{2} + C = \frac{-\cos(2x)}{4} + C$

   c) $\int(1+2x)^3dx = \left|\begin{array}{c} z = 1+2x \\ dz = 2dx \end{array}\right| = \int\frac{z^3}{2}dz = \frac{z^4}{8} + C = \frac{(1+2x)^4}{8} + C$

   d)

$$\int e^x\cdot 5^{x-1}dx = \frac{1}{5}\int(5e)^x dx = \frac{1}{5}\int e^{x\ln 5e}dx = \left|\begin{array}{c} z = x\ln 5e \\ dz = \ln 5e dx \end{array}\right| = \frac{1}{5\ln 5e}\int e^z dz = \frac{1}{5\ln 5e}e^z + C$$
$$= \frac{1}{5\ln 5e}e^{x\ln 5e} + C = \frac{1}{5\ln 5e}(5e)^x + C$$

   d) $\int\frac{x}{|x|}dx = \begin{cases}\int 1dx = x + C, & x > 0\\ \int -1dx = -x + C, & x < 0\end{cases}$

   e) $\int x^5 dy = x^5 y + C$

5. a) $\int \frac{2}{\sqrt{1-4x^2}}dx = \left| \begin{array}{c} z = 2x \\ dz = 2dx \end{array} \right| = \int \frac{1}{\sqrt{1-z^2}}dz = \arcsin(z) + C = \arcsin(2x) + C$

b) $\int \frac{1}{x^2+9}dx = \int \frac{1}{9\left(\frac{x^2}{9}+1\right)}dx = \left| \begin{array}{c} z = \frac{x}{3} \\ dz = \frac{1}{3}dx \end{array} \right| = \int \frac{1}{3(z^2+1)}dz = \frac{1}{3}\mathrm{arctg}(z) + C = \frac{1}{3}\mathrm{arctg}\left(\frac{x}{3}\right) + C$

c) $\int \frac{1}{4x^2+3}dx = \int \frac{1}{3\left(\frac{4}{3}x^2+1\right)}dx = \left| \begin{array}{c} z = \frac{2x}{\sqrt{3}} \\ dz = \frac{2}{\sqrt{3}}dx \end{array} \right| = \frac{1}{2\sqrt{3}}\int \frac{1}{z^2+1}dz = \frac{1}{2\sqrt{3}}\mathrm{arctg}(z) + C = \frac{1}{2\sqrt{3}}\mathrm{arctg}(\frac{2x}{\sqrt{3}}) + C$

d) $\int x\sqrt{x^2-4}dx = \left| \begin{array}{c} z = x^2-4 \\ dz = 2xdx \end{array} \right| = \frac{1}{2}\int \sqrt{z}dz = \frac{z^{\frac{3}{2}}}{3} + C = \frac{(x^2-4)^{\frac{3}{2}}}{3} + C$

e) $\int \frac{e^x}{1+e^x}dx = \left| \begin{array}{c} z = 1+e^x \\ dz = e^x dx \end{array} \right| = \int \frac{1}{z}dz = \ln|z| + C = \ln(1+e^x) + C$

f) $\int \frac{1}{1+e^{-x}}dx = \int \frac{e^x}{e^x}\cdot\frac{1}{1+e^{-x}}dx = \int \frac{e^x}{1+e^x}dx = \ln(1+e^x) + C$

g) $\int \frac{3^x}{1+9^x}dx = \left| \begin{array}{c} z = 3^x \\ dz = 3^x \ln 3 dx \end{array} \right| = \frac{1}{\ln 3}\int \frac{1}{1+z^2}dz = \frac{1}{\ln 3}\mathrm{arctg}(z) + C = \frac{1}{\ln 3}\mathrm{arctg}(3^x) + C$

h) $\int \frac{1}{x^2+4x+5}dx = \int \frac{1}{(x+2)^2+1}dx = \left| \begin{array}{c} z = x+2 \\ dz = dx \end{array} \right| = \int \frac{1}{z^2+1}dz = \mathrm{arctg}(z) + C = \mathrm{arctg}(x+2) + C$

6. a) $\int \mathrm{tg}^2 x dx = \left| \begin{array}{c} z = \mathrm{tg}x \\ x = \mathrm{arctg}z \\ dx = \frac{1}{1+z^2}dz \end{array} \right| = \int \frac{z^2}{1+z^2}dz = \int \left(1 - \frac{1}{1+z^2}\right)dz = z - \mathrm{arctg}(z) + C = \mathrm{tg}x - x + C$

b) $\int \frac{\sin 2x}{5\cos x}dx = \int \frac{2\sin x\cos x}{5\cos x}dx = \frac{2}{5}\int \sin x dx = -\frac{2}{5}\cos x + C$

c)

$$\int \sin^2\frac{x}{2} \overset{p.p.}{=} -2\sin\frac{x}{2}\cos\frac{x}{2} + \int \cos^2\frac{x}{2}dx = -2\sin\frac{x}{2}\cos\frac{x}{2} + \int (1-\sin^2\frac{x}{2})dx$$
$$= -2\sin\frac{x}{2}\cos\frac{x}{2} + x - \int \sin^2\frac{x}{2}dx = -\sin\frac{x}{2}\cos\frac{x}{2} + \frac{x}{2} + C$$

d) $\int \frac{\cos 2x}{\sin^2 x\cos^2 x}dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\cos^2 x}dx = -\mathrm{cotg}x - \mathrm{tg}x + C$

e) $\int \mathrm{cotg}x dx = \int \frac{\cos x}{\sin x}dx = \left| \begin{array}{c} z = \sin x \\ dz = \cos x dx \end{array} \right| = \int \frac{1}{z}dz = \ln|z| + C = \ln|\sin x| + C$

f) $\int \sin^3 x dx = \int \sin x(1-\cos^2 x)dx = \left| \begin{array}{c} z = \cos x \\ dz = -\sin x dx \end{array} \right| = -\int (1-z^2)dz = -z + \frac{z^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C$

g)

$$\int \sqrt{1-x^2}dx = \left| \begin{array}{c} x = \sin z \\ dx = -\cos z dz \end{array} \right| = \int \sqrt{1-\sin^2 z}\cos z dz = \int \cos^2 z dz \overset{p.p.}{=} \sin z\cos z + \int \sin^2 z dz$$
$$= \sin z\cos z + \int (1-\cos^2 z)dz = \sin z\cos z + z - \int \cos^2 z dz = \frac{\sin z\cos z + z}{2} + C$$
$$= \frac{\sin z\sqrt{1-\sin^2 z} + z}{2} + C = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C$$

h) $\int x^2\sqrt{x^3+5}dx = \left| \begin{array}{c} z = x^3+5 \\ dz = 3x^2 dx \end{array} \right| = \frac{1}{3}\int \sqrt{z}dz = \frac{2}{9}z^{\frac{3}{2}} + C = \frac{2}{9}(x^3+5)^{\frac{3}{2}} + C$

i) $\int x^2\sin x dx \overset{p.p.}{=} -x^2\cos x + \int 2x\cos x dx \overset{p.p.}{=} -x^2\cos x + 2x\sin x - 2\int \sin x dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

j) $\int \cos^3 x\sin x dx = \left| \begin{array}{c} z = \cos x \\ dz = -\sin x dx \end{array} \right| = -\int z^3 dz = \frac{-z^4}{4} + C = \frac{-\cos^4 x}{4} + C$

k) $\int \frac{\ln x}{x}dx = \left| \begin{array}{c} z = \ln x \\ dz = \frac{1}{x}dx \end{array} \right| = \int z dz = \frac{z^2}{2} + C = \frac{\ln^2 x}{2} + C$

l) $\int \text{arcsin}x dx \overset{p.p.}{=} x\text{arcsin}x - \int x\frac{1}{\sqrt{1-x^2}}dx = x\text{arcsin}x + \sqrt{1-x^2} + C$

$$\int x\frac{1}{\sqrt{1-x^2}}dx = \left| \begin{array}{c} z = 1-x^2 \\ dz = -2xdx \end{array} \right| = \frac{-1}{2}\int z^{-\frac{1}{2}} = -z^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C$$

m) $\int \sin x \cdot \ln \text{tg}x dx \overset{p.p.}{=} -\cos x \cdot \ln \text{tg}x + \int \frac{1}{\sin x}dx = -\cos x \cdot \ln \text{tg}x + \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$

$$\int \frac{1}{\sin x}dx = \int \frac{\sin x}{1-\cos^2 x}dx = \left| \begin{array}{c} z = \cos x \\ dz = -\sin x dx \end{array} \right| = \int \frac{1}{z^2-1}dz = \frac{1}{2}\int\left(\frac{1}{z-1} + \frac{-1}{z+1}\right)$$
$$= \frac{1}{2}\left(\ln|z-1| - \ln|z+1|\right) + C = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

n)

$$\int \ln(x+\sqrt{1+x^2})dx \overset{p.p.}{=} x\ln(x+\sqrt{1+x^2}) - \int x\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)dx$$
$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}dx = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

$$\int \frac{x}{\sqrt{1+x^2}}dx = \left| \begin{array}{c} z = 1+x^2 \\ dz = 2xdx \end{array} \right| = \frac{1}{2}\int \frac{1}{\sqrt{z}}dz = \sqrt{z} + C = \sqrt{1+x^2} + C$$

o) $\int \frac{\ln \sin x}{\sin^2 x}dx \overset{p.p.}{=} -\text{cotg}x \ln \sin x + \int \text{cotg}x\frac{\cos x}{\sin x}dx = -\text{cotg}x \ln \sin x + \int \text{cotg}^2 x dx = -\text{cotg}x \ln \sin x - \text{cotg}x - x + C$

$$\int \text{cotg}^2 x dx = \left| \begin{array}{c} z = \text{tgx} \\ \frac{1}{1+z^2}dz = dx \end{array} \right| = \int \frac{1}{z^2(1+z^2)}dz \overset{(*)}{=} \int\left(\frac{1}{z^2} - \frac{1}{1+z^2}\right)dz = -\frac{1}{z} - \text{arctg}(z) + C$$
$$= -\text{cotg}x - x + C$$

$$\frac{1}{z^2(1+z^2)} \overset{(*)}{=} \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{1+z^2} = \frac{Az(1+z^2) + B(1+z^2) + Cz^3 + Dz^2}{z^2(1+z^2)}$$

$$1 = (A+C)z^3 + (B+D)z^2 + Az + B$$
$$B = 1$$
$$A = 0$$
$$C = 0$$
$$D = -1$$

p)

$$\int \sin x \cdot \sin 2x \cdot \sin 3x dx = \int (6\sin^3 x \cos^3 x - 2\sin^5 x \cos x)dx = \int [6\sin^3 x(1-\sin^2 x) - 2\sin^5 x]\cos x dx$$
$$\left| \begin{array}{c} z = \sin x \\ dz = \cos x dx \end{array} \right| = \int (6z^3 - 8z^5)dz = \frac{3z^4}{2} - \frac{4z^6}{3} + C = \frac{3\sin^4 x}{2} - \frac{4\sin^6 x}{3} + C$$

7. a) $\int(\frac{5}{3x-2} + \frac{1}{(1-4x)^2})dx = \frac{5}{3}\ln|3x-2| + \frac{1}{4(1-4x)} + C$

b) $\int \frac{2x-3}{x^2-3x+5}dx = \left| \begin{array}{c} z = x^2-3x+5 \\ dz = (3x-2)dx \end{array} \right| = \int \frac{1}{z}dz = \ln|z| + C = \ln|x^2-3x+5| + C$

c) $\int \frac{1}{x^2-2x+5}dx = \int \frac{1}{(x-1)^2+4}dx = \frac{1}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}dx = \left| \begin{array}{c} z = \frac{x-1}{2} \\ dz = \frac{1}{2}dx \end{array} \right| = \frac{1}{2}\int \frac{1}{z^2+1}dz = \frac{1}{2}\text{arctg}z + C = \frac{1}{2}\text{arctg}\frac{x-1}{2} + C$

d) $\int \frac{x-4}{2x^2-x+2}dx = \frac{1}{4}\int \frac{4x-1}{2x^2-x+2}dx - \frac{15}{4}\int \frac{1}{2x^2-x+2}dx = \frac{1}{4}\ln|x^2-x+2| - \frac{\sqrt{15}}{2}\text{arctg}\frac{4x-1}{\sqrt{15}} + C$

$$\int \frac{4x-1}{2x^2-x+2}dx = \left| \begin{array}{c} z = 2x^2-x+2 \\ dz = (4x-1)dx \end{array} \right| = \int \frac{1}{z}dz = \ln|z| + C = \ln|x^2-x+2| + C$$

$$\int \frac{1}{2x^2-x+2}dx = \frac{1}{2}\int \frac{1}{(x-\frac{1}{4})^2 + \frac{15}{16}} = \frac{8}{15}\int \frac{1}{\frac{(x-\frac{1}{4})^2}{\frac{15}{16}} + 1}dx = \frac{8}{15}\int \frac{1}{\left(\frac{4x-1}{\sqrt{15}}\right)^2 + 1}dx = \left| \begin{array}{c} z = \frac{4x-1}{\sqrt{15}} \\ dz = \frac{4}{\sqrt{15}}dx \end{array} \right| =$$
$$= \frac{2}{\sqrt{15}}\int \frac{1}{z^2+1}dz = \frac{2}{\sqrt{15}}\text{arctg}z + C = \frac{2}{\sqrt{15}}\text{arctg}\frac{4x-1}{\sqrt{15}} + C$$

e) $\int \frac{1}{x^2+2x}dx = \frac{1}{2}\int\left(\frac{1}{x}+\frac{-1}{x+2}\right)dx = \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$

f)

$$\int \frac{x^4}{x^4+5x^2+4}dx = \int\left(1-\frac{5x^2+4}{x^4+5x^2+4}\right)dx \overset{(*)}{=} x+\frac{1}{3}\int\left(-\frac{1}{x^2+1}+\frac{16}{x^2+4}\right)dx = x+\frac{1}{3}\int\left(-\frac{1}{x^2+1}+\frac{4}{\left(\frac{x}{2}\right)^2+1}\right)$$

$$= x+-\frac{1}{3}\mathrm{arctg}x+\frac{8}{3}\mathrm{arctg}\frac{x}{2}+C$$

$$\frac{5x^2+4}{x^4+5x^2+4} \overset{(*)}{=} \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)}$$

$$= \frac{(A+C)x^3+(B+D)x^2+(4A+C)x+4B+D}{(x^2+1)(x^2+4)}$$

$$5x^2+4 = (A+C)x^3+(B+D)x^2+(4A+C)x+4B+D$$

$$A = 0$$

$$C = 0$$

$$B = -\frac{1}{3}$$

$$D = \frac{16}{3}$$

g) $\int(\frac{x}{x^2-3x+2})^2dx = \int\frac{x^2}{(x-2)^2(x-1)^2}dx \overset{(*)}{=} \int\left(-4\frac{1}{x-2}+4\frac{1}{(x-2)^2}+4\frac{1}{(x-1)}+\frac{1}{(x-1)^2}\right)dx = 4\ln\left|\frac{x-1}{x-2}\right| - \frac{4}{x-2}-\frac{1}{x-1}+C$

$$\frac{x^2}{(x-2)^2(x-1)^2} \overset{(*)}{=} \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-1)}+\frac{D}{(x-1)^2}$$

$$= \frac{A(x-2)(x-1)^2+B(x-1)^2+C(x-2)^2(x-1)+D(x-2)^2}{(x-2)^2(x-1)^2}$$

$$1 = D$$

$$4 = B$$

$$0 = A+C$$

$$0 = -2A+B-4C+4D$$

$$C = 4$$

$$A = -4$$

h)

$$\int\frac{x^3}{(x-1)(x-2)^2}dx = \int\frac{x^3}{x^3-3x^2+6x-4}dx = \int\left(1+\frac{3x^2-6x+4}{(x-1)(x-2)^2}\right) \overset{(*)}{=} \int\left(1+\frac{1}{x-1}+\frac{2}{x-2}+\frac{4}{(x-2)^2}\right)dx$$

$$= x+\ln|x-1|+2\ln|x-2|-\frac{4}{x-2}+C$$

$$\frac{3x^2-6x+4}{(x-1)(x-2)^2} \overset{(*)}{=} \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2} = \frac{A(x-2)^2+B(x-1)(x-2)+C(x-1)}{(x-1)(x-2)^2}$$

$$1 = A$$

$$4 = C$$

$$3 = A+B$$

$$2 = B$$

8. a) $\int\frac{1}{1-\cos x}dx = \left|\begin{array}{c} t=\mathrm{tg}\frac{x}{2} \\ \frac{2}{1+t^2}dt = dx\end{array}\right| = \int\frac{1}{1-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt = \int\frac{1}{t^2}dt = -\frac{1}{t}+C = -\frac{1}{\mathrm{tg}\frac{x}{2}}+C = -\mathrm{cotg}\frac{x}{2}+C$

b) $\int\cos^3 x dx = \int(1-\sin^2 x)\cos x dx = \left|\begin{array}{c} t=\sin x \\ dt=\cos x dx\end{array}\right| = \int(1-t^2)dt = t-\frac{t^3}{3}+C = \sin x - \frac{\sin^3 x}{3}+C$

c) $\int \frac{1}{\cos^4 x}dx = \left| \begin{array}{c} t = \mathrm{tg}x \\ \frac{1}{1+t^2}dt = dx \end{array} \right| = \int \frac{1}{\left(\frac{1}{1+t^2}\right)^2} \cdot \frac{1}{1+t^2}dt = \int (1+t^2)dx = t + \frac{t^3}{3} + C = \mathrm{tg}x + \frac{\mathrm{tg}^3 x}{3} + C$

d)

$$\int \mathrm{tg}^2 x - \mathrm{tg}x dx = \left| \begin{array}{c} t = \mathrm{tg}x \\ \frac{1}{1+t^2}dt = dx \end{array} \right| = \int \frac{t^2 - t}{1+t^2}dt = \int \left(1 - \frac{t}{1+t^2} - \frac{1}{1+t^2}\right) dt = t - \frac{1}{2}\ln|1+t^2| - \mathrm{arctg}t + C$$
$$= \mathrm{tg}x - \ln\sqrt{1+\mathrm{tg}^2 x} - x + C$$

e)

$$\int \frac{1}{\sin^2 x - \sin x \cos x - 2\cos^2 x}dx = \left| \begin{array}{c} t = \mathrm{tg}x \\ \frac{1}{1+t^2}dt = dx \end{array} \right| = \int \frac{1}{\frac{t^2}{1+t^2} - \sqrt{\frac{t^2}{1+t^2}} \cdot \sqrt{\frac{1}{1+t^2}} - 2\frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}dt$$
$$= \int \frac{1}{t^2 - t - 2}dt = \int \frac{1}{(t+1)(t-2)}dt \overset{(*)}{=} \frac{1}{3}\int \left(\frac{-1}{t+1} + \frac{1}{t-2}\right) dt$$
$$= \frac{1}{3}\left(-\ln|t+1| + \ln|t-2|\right) + C = \frac{1}{3}\ln\frac{|\mathrm{tg}x - 2|}{|\mathrm{tg}x + 1|} + C$$
$$\frac{1}{(t+1)(t-2)} \overset{(*)}{=} \frac{A}{t+1} + \frac{B}{t-2} = \frac{A(t-2) + B(t+1)}{(t-1)(t+2)}$$
$$0 = A + B$$
$$1 = -2A + B$$
$$\frac{1}{3} = B$$
$$-\frac{1}{3} = A$$

f)

$$\int \frac{2\sin x - 3\cos x}{\sin x + 5\cos x}dx = \left| \begin{array}{c} t = \mathrm{tg}x \\ \frac{1}{1+t^2}dt = dx \end{array} \right| = \int \frac{\frac{2t}{\sqrt{1+t^2}} - \frac{3}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + \frac{5}{\sqrt{1+t^2}}} \cdot \frac{1}{1+t^2}dt = \int \frac{2t-3}{(t+5)(t^2+1)}dt$$
$$\overset{(*)}{=} \frac{1}{2}\int \left(\frac{-1}{t+5} + \frac{t-1}{t^2+1}\right) dt = \frac{-1}{2}\ln|t+5| + \frac{1}{4}\ln|t^2+1| - \frac{1}{2}\mathrm{arctg}t + C$$
$$= \frac{-1}{2}\ln|\mathrm{tg}x + 5| + \frac{1}{4}\ln|\mathrm{tg}x^2 + 1| - \frac{x}{2} + C$$
$$\frac{2t-3}{(t+5)(t^2+1)} \overset{(*)}{=} \frac{A}{t+5} + \frac{Bt+C}{t^2+1} = \frac{A(t^2+1) + (Bt+C)(t+5)}{(t^2+1)(t+5)}$$
$$0 = A + B$$
$$2 = 5B + C$$
$$-3 = A + 5C$$
$$\frac{1}{2} = B$$
$$-\frac{1}{2} = A$$
$$-\frac{1}{2} = C$$

9. a)

$$\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx = \int \frac{1-\sqrt{\frac{x-1}{x+1}}}{1+\sqrt{\frac{x-1}{x+1}}}dx = \left| \begin{array}{c} t=\sqrt{\frac{x-1}{x+1}}, t^2=\frac{x-1}{x+1} \\ x=\frac{t^2+1}{t^2-1}, dx=\frac{-4t}{(t^2-1)^2}dt \end{array} \right| = \int \frac{-4t(1-t)}{(1+t)(t^2-1)^2}dt$$

$$= \int \frac{4t}{(1+t)^3(t-1)}dt \overset{(*)}{=} \int \left(\frac{-\frac{1}{2}}{1+t} + \frac{-1}{(1+t)^2} + \frac{2}{(1+t)^3} + \frac{\frac{1}{2}}{t-1} \right) dt$$

$$= -\frac{1}{2}\ln|1+t| + \frac{1}{1+t} - \frac{1}{(1+t)^2} + \frac{1}{2}\ln|1-t| + C$$

$$= \ln\sqrt{\frac{1-t}{1+t}} + \frac{t}{(1+t)^2} + C = \ln\sqrt{\frac{1-\sqrt{\frac{x-1}{x+1}}}{1+\sqrt{\frac{x-1}{x+1}}}} + \frac{\sqrt{\frac{x-1}{x+1}}}{(1+\sqrt{\frac{x-1}{x+1}})^2} + C$$

$$\frac{4t}{(1+t)^3(t-1)} \overset{(*)}{=} \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{(1+t)^3} + \frac{D}{t-1}$$

$$= \frac{A(1+t)^2(t-1)+B(1+t)(t-1)+C(t-1)+D(1+t)^3}{(t+1)^3(t-1)}$$

$$\begin{aligned} 4 &= 8D \\ \frac{1}{2} &= D \\ -4 &= -2C \\ 2 &= C \\ 0 &= A+D \\ -\frac{1}{2} &= A \\ 0 &= -A-B-C+D \\ -1 &= B \end{aligned}$$

b)

$$\int \sqrt{\frac{a+x}{a-x}}dx = \left| \begin{array}{c} t=\sqrt{\frac{a+x}{a-x}}, t^2=\frac{a+x}{a-x} \\ x=\frac{a(t^2-1)}{t^2+1}, dx=\frac{4at}{(t^2+1)^2}dt \end{array} \right| = \int \frac{4at^2}{(t^2+1)^2}dt \overset{p.p.}{=} 2a\left[t\cdot\frac{-1}{(t^2+1)} + \int \frac{1}{t^2+1}dt\right]$$

$$= 2a\left[\frac{-t}{(t^2+1)} + \mathrm{arctg}t\right] + C = 2a\left[\frac{-\sqrt{\frac{a+x}{a-x}}}{\frac{a+x}{a-x}+1} + \mathrm{arctg}\sqrt{\frac{a+x}{a-x}}\right] + C$$

$$= -\sqrt{(x-a)(x+a)} + 2a\cdot\mathrm{arctg}\sqrt{\frac{a+x}{a-x}} + C$$

c)

$$\int \frac{1}{\sqrt{1-4x-x^2}}dx = \left| \begin{array}{c} \sqrt{1-4x-x^2}=1+tx, 1-4x-x^2=1+2xt+t^2x^2 \\ x=\frac{-2(2+t)}{t^2+1}, dx=\frac{2(t^2+4t-1)}{(t^2+1)^2}dt \end{array} \right| =$$

$$= \int \frac{1}{1+t\cdot\frac{-2(2+t)}{t^2+1}}\cdot\frac{2(t^2+4t-1)}{(t^2+1)^2}dt = \int \frac{-2}{t^2+1}dt = -2\mathrm{arctg}t + C$$

$$= -2\mathrm{arctg}\left(\frac{\sqrt{1-4x-x^2}-1}{x}\right) + C$$

d)

$$\int \frac{1}{\sqrt{1+x+x^2}}dx = \left| \begin{array}{c} \sqrt{1+x+x^2} = t+x, 1+x+x^2 = t^2+2tx+x^2 \\ x = \frac{t^2-1}{1-2t}, dx = \frac{2(t-t^2-1)}{(1-2t)^2}dt \end{array} \right| = \int \frac{1}{t+\frac{t^2-1}{1-2t}} \cdot \frac{2(t-t^2-1)}{(1-2t)^2}dt$$
$$= \int \frac{2}{1-2t}dt = -\ln|1-2t| + C = -\ln|1-2(\sqrt{1+x+x^2}-x)| + C$$

e) $\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{1-(\frac{2}{3}x)^2}}dx = \left| \begin{array}{c} t = \frac{2}{3}x \\ dt\frac{3}{2} = dx \end{array} \right| = \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}dt = \frac{1}{2}\arcsin(t) + C = \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C$

f)

$$\int \frac{1}{x\sqrt{1+x^2}}dx = \left| \begin{array}{c} t = \sqrt{1+x^2} \\ dt = \frac{x}{\sqrt{1+x^2}}dx \end{array} \right| = \int \frac{1}{t^2-1}dt = \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \ln\sqrt{\frac{t-1}{t+1}} + C$$
$$= \ln\sqrt{\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}} + C$$

10. a) $\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}}\frac{1}{1+x}dx = \left| \begin{array}{c} t = \operatorname{arctg}\sqrt{x} \\ dt = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}dx \end{array} \right| = \int 2t dt = t^2 + C = \operatorname{arctg}^2\sqrt{x} + C$

b)

$$\int \frac{\cos x}{\sqrt{1+\sin^2 x}}dx = \left| \begin{array}{c} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \frac{1}{\sqrt{1+t^2}}dt = \left| \begin{array}{c} \sqrt{1+t^2} = t+z, 1+t^2 = t^2+2tz+z^2 \\ t = \frac{1-z^2}{2z}, dt = -\frac{z^2+1}{2z^2}dz \end{array} \right|$$
$$= \int \frac{-(z^2+1)}{(\frac{1-z^2}{2z}+z)2z^2}dz = \int \frac{-1}{z}dz = -\ln|z| + C = -\ln|\sqrt{1+t^2}-t| + C$$
$$= -\ln|\sqrt{1+\sin^2 x} - \sin x| + C$$

c) $\int \frac{e^x}{e^{2x}-1}dx = \left| \begin{array}{c} t = e^x \\ dt = e^x dx \end{array} \right| = \int \frac{1}{t^2-1}dt = \frac{1}{2}\int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$

d)

$$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx = \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\right)dx = \arcsin x + \int \frac{1}{\sqrt{1+x^2}}dx$$
$$\overset{viz 10.b)}{=} \arcsin x - \ln|\sqrt{1+x^2}-x| + C$$

e)

$$\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx = \int \left(\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2+1}}\right)dx = -\ln|\sqrt{x^2-1}-x| - \ln|\sqrt{1+x^2}-x| + C$$
$$\int \frac{1}{\sqrt{x^2-1}}dx = \left| \begin{array}{c} \sqrt{x^2-1} = x+t, -1 = 2xt+t^2 \\ x = -\frac{1+t^2}{2t}, dx = -\frac{2(t^2-1)}{4t^2}dt \end{array} \right| = \int \frac{-2(t^2-1)}{\left(-\frac{1+t^2}{2t}+t\right)4t^2}dt = \int \frac{-1}{t}dt = -\ln|t| + C$$
$$= -\ln|\sqrt{x^2-1}-x| + C$$

f) $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}dx = \int (1-x)^{\frac{-3}{5}}dx = \frac{-5}{2}(1-x)^{\frac{2}{5}} + C$

g) $\int \frac{1}{\sin^2(2x+\frac{\pi}{4})}dx = \left| \begin{array}{c} t = 2x+\frac{\pi}{4} \\ dt = 2dx \end{array} \right| = \frac{1}{2}\int \frac{1}{\sin^2 t}dt = -\frac{1}{2}\operatorname{cotg}t + C = \frac{1}{2}\operatorname{cotg}(2x+\frac{\pi}{4}) + C$

h) $\int \frac{x^3}{x^8-2}dx = \left| \begin{array}{c} t = x^4 \\ dt = 4x^3 dx \end{array} \right| = \frac{1}{4}\int \frac{1}{t^2-2}dt = \frac{1}{8\sqrt{2}}\int\left(\frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\right)dt = \frac{1}{8\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C = \frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right| + C$

i) $\int \frac{1}{(1+x)\sqrt{x}}dx = \left| \begin{array}{c} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}dx \end{array} \right| = 2\int \frac{1}{1+t^2}dt = 2\operatorname{arctg}t + C = 2\operatorname{arctg}\sqrt{x} + C$

j) $\int \frac{1}{\sqrt{1+e^{2x}}}dx = \left| \begin{array}{c} t = e^x \\ dt = e^x dx \end{array} \right| = \int \frac{1}{t\sqrt{1+t^2}}dt \overset{viz 9.f)}{=} \frac{1}{2}\ln\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}dt + C = \frac{1}{2}\ln\frac{\sqrt{1+e^{2x}}-1}{\sqrt{1+e^{2x}}+1}dt + C$

k) $\int \frac{1}{x\ln x\ln(\ln x)}dx = \ln(\ln(\ln x))) + C$

l) $\int \frac{1}{x\sqrt{x^2-1}}dx = \left| \begin{array}{c} t = \sqrt{x^2-1} \\ dt = \frac{x}{\sqrt{x^2-1}}dx \end{array} \right| = \int \frac{1}{t^2+1}dt = \operatorname{arctg} t + C = \operatorname{arctg}\sqrt{x^2-1} + C$

m) $\frac{\sin x}{\sqrt{\cos^3 x}}dx = \left| \begin{array}{c} t = \cos x \\ dt = -\sin x dx \end{array} \right| = -\int \frac{1}{\sqrt{t^3}}dt = \frac{2}{\sqrt{t}} + C = \frac{2}{\sqrt{\cos x}} + C$

n)

$$\int \frac{\sin x\cos x}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}}dx = \int \frac{\sin x\cos x}{\sqrt{(a^2-b^2)\sin^2 x + b^2}}dx = \left| \begin{array}{c} t = (a^2-b^2)\sin^2 x + b^2 \\ dt = 2(a^2-b^2)\sin x\cos x dx \end{array} \right|$$

$$= \frac{1}{2(a^2-b^2)}\int \frac{1}{\sqrt{t}}dt = \frac{\sqrt{t}}{2(a^2-b^2)} + C = \frac{\sqrt{(a^2-b^2)\sin^2 x + b^2}}{2(a^2-b^2)} + C$$