Abelian groups with a minimal generating set

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“STA 2011"

September 6

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Notation

Given an algebra A, we denote by $\mathrm{Gen}(A)$ the minimal size of its generating set.

Lemma (A)

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Let A be an abelian group and let F be a free abelian group with a free basis X. If $\mathrm{Card}(X) \geq \mathrm{Card}(A)$, then $F \oplus A$ has a minimal generating set.

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Theorem (1)

Let D be a divisible abelian group and let A be an abelian group with a minimal generating. Then the direct sum $A \oplus D$ has a minimal generating set if and only if $\mathrm{Gen}(A) \geq \mathrm{Card}(D)$.

Lemma

Let A be an abelian group. If the dimensions of A/pA, and A/qA (as $\mathbb{Z}_p$, and $\mathbb{Z}_q$ vector space, respectively) are the same as $\mathrm{Gen}(A)$, for two different prime numbers p, q, then A possesses a minimal generating set.

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- $\mathbb{Q} \oplus \mathbb{Z}$ has no minimal generating set while

$$(\mathbb{Q} \oplus \mathbb{Z})^{(\aleph_0)} \simeq \mathbb{Q}^{(\aleph_0)} \oplus \mathbb{Z}^{(\aleph_0)}$$

  has.

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The group

$$P = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$$

does not have a minimal set of generators.

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- *Let A be a non-divisible abelian group of size λ. If $\varkappa$ is a cardinal number satisfying $\lambda^{\varkappa} = 2^{\varkappa}$, then $A^{\varkappa}$ has a minimal generating set.*
- *In particular, an infinite product of copies of a countable group A has a minimal generating set if and only if the group A is not divisible.*

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Example

Let p be a prime number. The torsion subgroup T of the product

$$\prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$$

is reduced and it does not have a minimal generating set.

Lemma

Let A be a torsion abelian group and B its basic subgroup. If $\mathrm{Card}(B) < \mathrm{Card}(A)$, then A does not have a minimal generating set.

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Lemma (*Recall*)

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Lemma (*Recall*)

Every countable reduced torsion group has a minimal generating set.

Theorem (3)

Let A be an infinite torsion abelian group with a basic subgroup B. Then A has a minimal generating set if and only if $\mathrm{Card}(B) = \mathrm{Card}(A)$.

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- A torsion free abelian group A of rank 1 is of Type I if its characteristic is represented by a height sequence $\boldsymbol{h} = (h_p)_{p \in \mathbb{P}}$ with $0 < h_p < \infty$ for infinitely many primes.
- A torsion free abelian group A of rank 1 is of Type II if it is neither divisible nor of Type I.

Lemma

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *be a finite direct sum of torsion-free abelian groups* $A_1, A_2, \ldots, A_n$ *of rank 1. Then* A *has a minimal generating set if and only if either* A *is free or one of the groups* A_i *is of Type I.*

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An infinite direct sum of torsion-free abelian groups of rank 1 of Type II has a minimal generating set.

Lemma (C)

Every completely decomposable reduced torsion free abelian group of an infinite rank has a minimal generating set.

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Theorem (4)

Let A be a completely decomposable torsion free abelian group of an infinite rank and let $A = R \oplus D$ be its decomposition into a direct sum of a divisible group D and a reduced abelian group R. Then A has a minimal generating set if and only if $\mathrm{Rank}(R) \geq \mathrm{Card}(D)$.