Summer School on Algebra and Ordered Sets
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ABSTRACTS
Invited talks

Forks, finitely related clones, and finitely generated varieties

Erhard Aichinger
Johannes Kepler University Linz
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We will present the proofs of the following two results:
(1) A clone with edge term on a finite set is finitely related (Aichinger, Mayr, McKenzie, 2009).
(2) Every subvariety of a finitely generated variety with edge term is finitely generated (Aichinger, Mayr, 2014).

For proving the first item, we use forks to represent the functions in the clone of a fixed arity. We exploit one connection between these forks, for different arities, to exclude infinite descending chains of clones.

Geometric and topological tools for MV-algebras and unital \(l\)-groups

Daniele Mundici
University of Florence

Unital lattice ordered abelian groups (unital \(l\)-groups, for short) provide a modern formalization of the time honored theory of euclidean magnitudes with a distinguished archimedean unit of measurement (Hölder representation). MV-algebras stand to boolean algebras as Lukasiewicz infinite-valued logic stands to boolean logic (Chang completeness). There is a natural equivalence \(\Gamma\) between unital \(l\)-groups and MV-algebras. Free objects in these two equivalent categories consist of piecewise linear functions with integer coefficients (McNaughton representation). Finitely presented objects are dually equivalent to
rational polyhedra in euclidean space (essentially, Baker-Beynon duality). Re-
cent developments in the theory of MV-algebras yield self-contained, pictorial
proofs of these fundamental results, showing their deep relationships with other
mathematical structures like combinatorial fans and approximately finite dimen-
sional (AF) $C^*$-algebras.

References
[1] D. Mundici, Introducing MV-algebras. These handouts will be made available
during the tutorial. They are based on:

Well quasi order in combinatorics and algebra

Nik Ruškuc
University of St. Andrews

The notion of well quasi-order (wqo) from the theory of ordered sets often
arises naturally in contexts where one deals with infinite collections of mathemat-
ical structures which can somehow be compared, and it then represents a useful
discriminator between 'tame' and 'wild' such classes. In my talks I will adopt the
viewpoint of algebra and combinatorics being dual counterpart parts of each other:
while algebra is concerned with sets equipped with some operations, many com-
binatorial objects (such as graphs, tournaments, posets and permutations) can
be viewed as relational structures. One is then often concerned about some
comparison relation between the objects under consideration – be it substruct-
ures, subalgebras or homomorphic images, and the question arises whether the
resulting ordering is wqo. I will attempt to review the main trends in this
way of thinking, and encourage the dialogue between different areas. The spe-
cific themes will include: definition and main examples, Higman's and Kruskal's
theorems, wqo classes of graphs under inclusion, graph minors and Robertson-
Seymour theorem, wqo in the theory of pattern avoidance of permutations with
enumeration and computational consequences, and a review of applications of
wqo in algebra.
Computer-Understandable Mathematics: Is It Coming?

**Josef Urban**
Czech Technical University in Prague

The talks will discuss development of computer-understandable (formal) mathematics in proof assistants and the recent advances in this field. We will briefly mention some recently finished large formalizations and systems such as the formal proof of the Kepler conjecture done in the HOL Light and Isabelle systems and the formal proof of the Odd Order theorem done in the Coq system. Then we will describe and demonstrate some recent formalization technologies, such as linking of formal proof assistants to strong automated theorem provers and methods for bridging large libraries and proof developments. The final part will consist of more open-ended ideas such as the possibility of parsing informal math by combining linguistic and theorem-proving approaches, and various uses of state-of-the-art AI techniques over large math corpora.
Contributed talks

Modularity in Hyperlattice

Reza Ameri
University of Tehran
Coauthor: Mohsen Amiri

In this paper by considering the notion of a hyperlattice \( L \), we give some basic properties of them. We will proceed to introduce the notion of modular elements in a hyperlattice and use them to study of modular hyperlattices. We construct the pentagon hyperlattice \( N \) and show that every hyperlattice that have a copy of \( N_5 \) is non-modular, and show that the converse is not necessarily valid for hyperlattices. Also, we consider the fundamental relation on \( L \), as the smallest equivalence relation on \( L \) such that its quotient space is a modular lattice. Finally we construct fundamental functor from category of (resp. modular) hyperlattices to the category of (resp. modular) lattices. Acknowledgements

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(Strong totally) L-ordered Groups

R. A. Borzooei
Shahid Beheshti University
Coauthors: F. Hosseini, O. Zahiri

In this paper, first, by the concept of join and meet on an L-ordered set, the notion of an L-lattice is introduced and some related results are obtained. Then we applied them to define an L-ordered group. Moreover, we investigate some properties of L-ordered groups and then we introduce the notions of totally L-ordered group and strong totally L-ordered group, where \( L \) is a frame. Finally, we show that if \( (T, e) \) is a totally L-ordered group, then \( B(T) \), the set of all
isomorphisms on $T$, forms an $L$-ordered group. Moreover, if $(T, e)$ is a strong totally $L$-ordered group, then $B(T)$ is an $L$-lattice ordered group.

Subdirectly irreducible commutative multiplicatively idempotent semirings

Ivan Chajda
Palacky University Olomouc
Coauthor: Helmut Länger

It was shown by F. Guzmán that there are only two subdirectly irreducible members in the variety of Boolean semirings, i.e. the variety of multiplicatively idempotent semirings satisfying the identity $1 + x + x = 1$. On the contrary, we will show that there exists a proper class of subdirectly irreducible members in the variety $V$ of commutative idempotent semirings and, moreover, for every cardinal $k$ there exists at least two subdirectly semirings in $V$ whose cardinality is equal to $k$.

Geometric constructibility of cyclic polygons

Gábor Czédli
University of Szeged
Coauthor: Ádám Kunos

We study convex cyclic polygons, that is, inscribed $n$-gons. Starting from P. Schreiber’s idea from 1993, we prove that these polygons are not constructible from their side lengths with straightedge and compass, provided $n > 4$. They are non-constructible even in the particular case where they only have two different integer side lengths, provided $n > 5$. For $n$ even, we have an elementary proof. Otherwise, we had to develop two theorems of separate interest. Our Limit Theorem states that geometric constructibility is preserved under taking limits, while the Rational Parameter Theorem turns a non-constructibility result with a transcendental parameter into a non-constructibility result with a rational parameter; only under reasonable conditions in both theorems, of course.
Pseudocomplementation and special elements in basic algebras

Petr Emanovský
Palacký University Olomouc
Coauthor: Jan Kühr

We deal with pseudocomplements and some special types of elements in bounded lattices with antitone involutions (which are equivalent to the so-called basic algebras). We characterize subdirectly irreducible monotone basic algebras with pseudocomplementation. We discuss the connections between central, boolean, sharp and distributive elements.

Monounary algebras with SEKP

Emília Halušková
Mathematical Institute SAS, Košice

Let $A$ be an algebra. If every congruence of $A$ is a kernel of some endomorphism of $A$, then it is said that an algebra $A$ has an endomorphism kernel property, shortly $A$ has EKP. Let $\varphi$ be an endomorphism of $A$. If $\varphi$ preserves all congruences of $A$, then $\varphi$ is strong called. If every congruence of $A$ is a kernel of some strong endomorphism of $A$, then it is said that an algebra $A$ has a strong endomorphism kernel property, shortly $A$ has SEKP. All monounary algebras with SEKP will be described. Further, some conditions concerning to monounary algebras with EKP will be presented.

On monoids in Rel

Gejza Jenča
Slovak University of Technology, Bratislava, Slovakia
Coauthor: Anna Jenčová

The category $\textbf{Rel}$ is the category of sets (objects) and relations (morphisms). Equipped with direct product of sets $\times$, $\textbf{Rel}$ is a monoidal category. Moreover, $\textbf{Rel}$ is a locally posetal 2-category, since every homset $\textbf{Rel}(A,B)$ is a poset with respect to inclusion. We examine the category of monoids in this category, with lax morphisms. This category includes, as subcategories, various interesting classes: hypermonoids, partial monoids, quantum logics and small categories. We show how the 2-categorical abstract nonsense gives rise to several previously defined notions in these categories, for example dimensional equivalence on effect algebras. This explains where do these definitions come from. Moreover, there are “new” notions (even for ordinary monoids!) coming
from the 2-structure on $\text{Rel}$ that appear to have interesting examples: the relation of divisibility on $(\mathbb{N}, +, 0)$ is an example of a monad in the 2-category of monoids in $\text{Rel}$.

\begin{quote}
\textbf{No $M_3$s and Maltsev implies majority}
\textsc{Alexandr Kazda}
\textsc{IST Austria}

Ross Willard has recently proved that if $J$ is an $(2,3)$-minimal instance of CSP such that the congruence lattice of each potato (set of all values of a given variable viewed as an algebra) does not have $M_3$ as a sublattice and $J$ has a (multisorted) Maltsev polymorphism, then $J$ has a solution. This is in contrast to the general situation, where Maltsev algebras give rise to CSP instances for which local consistency need not work. Ross then conjectured that perhaps every such $J$ must admit a multisorted majority operation, which would explain why local constency works in this case. We show that this is indeed the case even when $J$ is just $(1,1)$-minimal (ie. all constraints are subdirect).
\end{quote}

\begin{quote}
\textbf{Algebras assigned to ternary relations}
\textsc{Miroslav Kolařík}
\textsc{Palacký University Olomouc}
\textsc{Coauthors: Ivan Chajda, Helmut Länger}

We show that to every centred ternary relation $T$ on a set $A$ there can be assigned (in a non-unique way) a ternary operation $t$ on $A$ such that the identities satisfied by $(A; t)$ reflect relational properties of $T$. We classify ternary operations assigned to centred ternary relations and we show how the concepts of relational subsystems and homomorphisms are connected with subalgebras and homomorphisms of the assigned algebra $(A; t)$. We show that for ternary relations having a non-void median can be derived so-called median-like algebras $(A; t)$ which become median algebras if the median $M_T(a, b, c)$ is a singleton for all $a, b, c \in A$. Finally, we introduce certain algebras assigned to cyclically ordered sets.
\end{quote}
Divisibility in semigroup rings

MIROSLAV KORBELÁŘ
Czech Technical University in Prague, Faculty of Electrical Engineering

Finitely generated commutative rings with divisible additive part are trivial. It seems that this property may be extended to certain type of semigroup rings and help to solve a similar conjecture for semirings. We will discuss the case connected with small number of generators.

Matrix representations of ordered monoids

DAVID KRUML
Masaryk University

The natural completion functors sets → preordered sets → semilattices → complete semilattices are monoidal functors. As those they “preserve” monoidal and module objects. Since quantale modules (i.e. module objects in the category of complete semilattices) can be characterized by matrices, we get easily a similar description of modules for the other categories.

On the structure of e-cyclic residuated lattices

JAN KŮHR
Palacký University in Olomouc
Coauthors: Michal Botur, Constantine Tsinakis

We say that a residuated lattice (RL for short) is e-cyclic if it satisfies the equation \( x \setminus e \approx e / x \), which holds e.g. in \( \ell \)-groups, divisible RLs or integral RLs. We present some results on the lattice of convex subalgebras and then focus on Hamiltonian and normal valued e-cyclic RLs. We try to describe the structure of strictly simple e-cyclic RLs.
On an \( \rightarrow \)-Irreducibility in Heyting Lattices

Marcin Lazarz
Department of Logic and Methodology of Sciences, Wroclaw University, Poland
Coauthor: Krzysztof Siemienczuk

An element \( a \) of a Heyting lattice is called \( \rightarrow \)-irreducible if \( a = x \rightarrow y \) implies \( a = x \) or \( a = y \) (or equivalently, \( a = x \rightarrow y \) implies \( a = y \)). We prove the following

**Theorem.** An element \( a \) of a finite lattice \( L \) is \( \rightarrow \)-irreducible iff \( a \) is the least element in some maximal Boolean interval in \( L \).

An easy consequence of the theorem is a new characterization of the skeleton of a lattice: all \( \rightarrow \)-irreducible elements form the skeleton. Moreover, we ask whether an element can be presented as an \( \rightarrow \)-combination of \( \rightarrow \)-irreducible elements. If every element of a lattice has this property, then we say the lattice has \( \rightarrow \)-decomposition property. The problem we investigate is how to characterize lattices with the \( \rightarrow \)-decomposition property. In the talk we present some partial results and discuss selected conjectures.

Set representation of tense orthocomplemented posets

Jan Paseka
Department of Mathematics and Statistics, Masaryk University
Coauthor: Ivan Chajda

Propositional logics, both classical and non-classical, usually do not incorporate the dimension of time. However, already Aristotle mentioned that time plays an important role in evaluation of true values of propositions. This motivated a lot of authors to investigate the so-called Temporal logic, i.e., the logic where time is considered as a variable of the propositional formula. In what follows, we will submit algebraic tools for the axiomatization of tense operators on orthocomplemented posets. The first step to the axiomatization of tense operators then will be a construction using the time frame \((T, R)\). The second step is a solution of the representation problem in a case of tense Jauch-Piron orthocomplemented poset.
Homogenous counterexample for congruence modularity in idempotent varieties

Jakub Opršal
Charles University in Prague

Recently, Kearnes and Szendrei described cube term blockers in infinite idempotent algebras. We will discuss the similar for varieties that are not congruence modular. A counterexample for modularity is a triple of congruences of the same algebra that does not satisfy the modularity law. From a result by McGarry we know that in locally finite idempotent varieties there is a counterexample of a certain homogeneous form. We generalize this result to infinite algebras. One of the interesting corollaries of this fact is that whenever two idempotent varieties are not congruence modular, then so is their (interpretability) join.

Set representation of tense orthocomplemented posets

Jan Paseka
Department of Mathematics and Statistics, Masaryk University
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**Factorization of a digital image**

Jan Pavlík  
Brno University of Technology

We show that if a digital image is treated as a generalized ultrametric space, then a surjective mapping between such spaces which preserves distance and the scalar structure induces a segmentation of the image and this segmentation can be obtained via scalar action on the canonical generalized normed affine space. Some examples of this phenomenon in the digital geometry will be provided.

**Some notes on universal homogeneous morphisms**

Maja Pech  
TU Dresden & University of Novi Sad  
Coauthor: Christian Pech

Universal homogeneous morphisms arise as Fraïssé-limits of certain families of morphisms – naturally inhabiting a comma category. They were introduced to study universal objects in horn-equivalence classes of first order structures, as well as for studying algebraic and topological properties of the polymorphism clones of homogeneous structures such as the Rado graph or the countable random poset. Universal homogeneous morphisms tend to be somewhat mysterious objects of a fractal like nature. In this talk, our goal is to put them in a context with classical Fraïssé-theory, to speculate about particular instances and to make a connection with the by now classical notion of generic automorphisms and ample generic automorphisms.

**Lifting diagrams by Con functor**

Miroslav Ploščica  
Safarik University, Košice, Slovakia

Let $Con_c A$ denote the semilattice of compact congruences of an algebra $A$. The critical point between classes of algebras $V$ and $W$ is the smallest cardinality of a semilattice isomorphic to $Con_c A$ for some $A \in V$, but not for $A \in W$. In order to determine critical points we investigate the liftability of semilattice diagrams by the Con functor in $V$ and $W$.  

Atoms and coatoms in the lattice of congruence lattices of algebras on a (finite) set

Reinhard Pöschel
TU Dresden, Institut f. Algebra
Coauthors: Danica Jakubíková-Studenovská, Sándor Radeleczki

For a fixed (finite) set $A$ let $E$ denote the lattice of congruence lattices of algebras with base set $A$. We give an explicit description of the atoms and coatoms in $E$. In particular we indicate a set $F$ of unary functions on $A$ such that the coatoms of $E$ are exactly the congruence lattices of the form $\text{Con}(A, f)$ with $f \in F$.

Ideals and congruences in generalized pseudo effect algebras

Sylvia Pulmannova
Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia
Coauthors: D. Foulis, E. Vincekova

A generalized pseudo effect algebra (GPEA) is a partially ordered partial algebraic structure with a smallest element $0$, but not necessarily with a unit (i.e., a largest element). If a GPEA admits a so-called unitizing automorphism, then it can be embedded as an order ideal in its so called unitization, which has a unit. We study ideals and congruences in GPEAs and their behavior with respect their unitizations.

Distributivity for upper continuous and strongly atomic lattices

Krzysztof Siemieńczuk
Department of Logic and Methodology of Sciences, Wroclaw University, Poland
Coauthor: Marcin Łazarz

A lattice $L$ is said to be upper continuous if it is complete and $x \land \bigvee C = \bigvee \{ x \land c : c \in C \}$ holds for any $x \in L$ and for any chain $C \subseteq L$. A lattice $L$ is strongly atomic if $x < y$ implies $x \prec z \leq y$ for some $z \in L$. We consider the following strengthenings of Birkhoff Conditions:

(D) \hspace{1cm} (\forall x, y \in L)(x \land y < x, y \Rightarrow [x \land y, x \lor y] \cong B_2),

(D*) \hspace{1cm} (\forall x, y \in L)(x, y < x \lor y \Rightarrow [x \land y, x \lor y] \cong B_2),

where $B_2$ denotes a four-element Boolean lattice. Our main result is:
Theorem. If \( L \) is an upper continuous and strongly atomic lattice then \( L \) is distributive iff \( L \) satisfies \((D)\) and \((D^*)\).

In the talk we present the main idea of the proof. Moreover, we discuss some consequences of the theorem: (1) For an upper continuous and strongly atomic lattice upper and lower local distributivity imply distributivity; (2) If \( L \) is an upper continuous, strongly atomic, modular but non-distributive lattice then \( L \) contains a covering diamond.

Minimal generating system for Sylow 2-subgroup of alternating group.

Skuratovskii R. V.
PL NTU 'KPI' of Kiev, Ukraine

The aim of this paper is to research the structure of Sylow 2-subgroups and to construct a minimal generating system for such subgroups. In other words, the problem is not simply in the proof of existence of a generating set with elements for Sylow 2-subgroup of alternating group of degree and its constructive proof. For the construction of minimal generating set we used the representation of elements of group by automorphisms of portraits for binary tree. The main result is the proof of minimality of this generating set of the above described subgroups and also the description of their structure.

Algebra for efficient knot recognition

David Stanovský
Charles University, Prague & Kazakh-British Technical University, Almaty
Coauthors: Andrew Fish, Alexei Lisitsa

Given two knot diagrams, can one efficiently determine whether they represent the same knot (up to continuous manipulation with the rope, that is, ambient isotopy)? The answer is unknown. I will explain a recent promising method, so called quandle colorings, which uses certain non-associative binary algebras to color arcs of a knot. I will demonstrate that quandle coloring results in a practically efficient tool for knot recognition.
Solving linear AX-equations

Jiří Tůma
Charles University in Prague
Coauthor: D. Josěk

In cryptanalysis it is sometimes necessary to find all solutions of an equation formed by unknown vectors of bits of a fixed length, parameters, and operations of modular addition and xor. We present the beginnings of a general theory of such equations that allows to find a necessary and sufficient condition for solvability of such equations and to calculate the probability that randomly selected vectors satisfy the equations.