Further critical point between varieties of lattices

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2 September 2008

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- $\mathrm{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_m^{0,1}) = \aleph_2$ for all $n > m \geq 3$ (M. Ploščica, 2000)

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- We denote by $\mathrm{Id_c}(R)$ the set of all finitely generated two-sided ideals of R.

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The dimension monoid of a lattice

- Dim L is the commutative monoid defined by generators $\Delta(a, b)$, $a \leq b$ in L and relations
  $(D0)$ $\Delta(a, a) = 0$, for all $a \in L$
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- For $G \in \mathscr{P}$, put $\nabla(G) = G^+/{\asymp}$, where G^+ is the monoid of positive elements of G and $\asymp$ is the smalest congruence of M^+ such that $G^+/{\asymp}$ is a semilattice.

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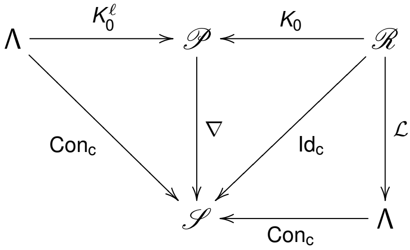
K_0, $\mathcal{L}$, $\mathrm{Id_c}$, K_0^ℓ, ∇ and $\mathrm{Con_c}$ are all functor, and the following diagram is commutative (up to natural equivalences) :



A lower bound of some critical point

Theorem

Let $\mathcal{V}$ be a variety of locally finite modular lattices. Let F be a field, let $n \in \mathbb{N}$ such that $\mathrm{lh}(K) \leq n$ for all K simple lattices of $\mathcal{V}$. Then :

$$\mathrm{crit}(\mathcal{V}; \mathbf{Var}(\mathrm{Sub}\, F^n)) \geq \aleph_2$$

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If $L \in \mathcal{V}$, such that card $L \leq \aleph_1$, then there exists R a regular ring, such that $\mathrm{Con_c}\, L \cong \mathrm{Con_c}(\mathcal{L}(R))$ and $\mathcal{L}(R) \in \mathbf{Var}(\mathrm{Sub}\, F^n)$.

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Moreover $\mathcal{L}(R_i) \in \mathbf{Var}(\mathrm{Sub}\, F^n)$ for all $i \in I$, Therefore $\mathcal{L}(\lim \vec{R}) \in \mathbf{Var}(\mathrm{Sub}\, F^n)$.
This can be extended to the unbounded case.

An upper bound of some critical points

Theorem

Let $\mathcal{V}$ be a finitely generated variety of lattices such that $M_n \notin \mathcal{V}$ and simple lattice of $\mathcal{V}$ are of length at most three, then:

$$\operatorname{crit}(\mathcal{M}_n; \mathcal{V}) \leq \aleph_2$$

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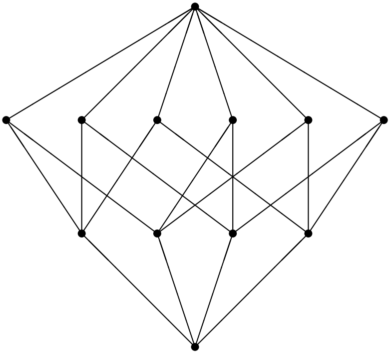
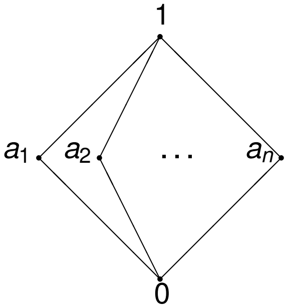


Figure: The poset I_4

A diagram of $\mathcal{M}_n$

The lattice M_n



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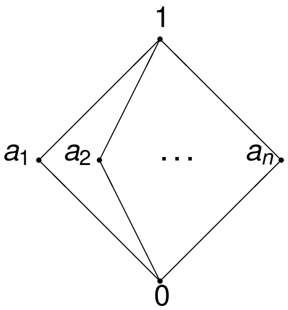

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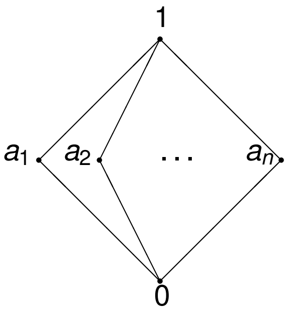

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is a direct system of $\mathcal{M}_n$.

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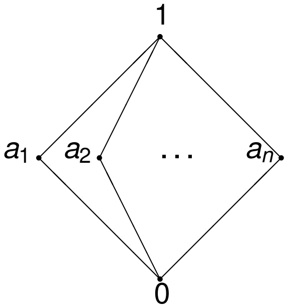

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- A poset U is *supported* if all finite subset can be extended to a kernel of U.
- $(U, |\cdot|)$ is a *norm-covering* of I if U is a supported poset and $|\cdot|\colon U \to I$ is isotone.

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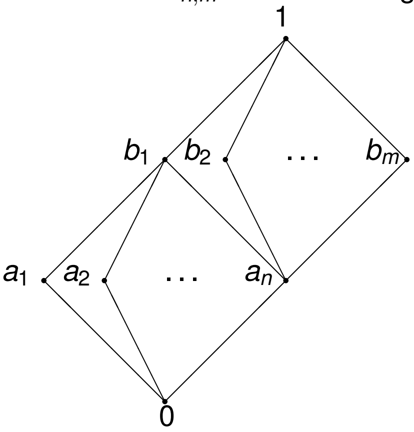

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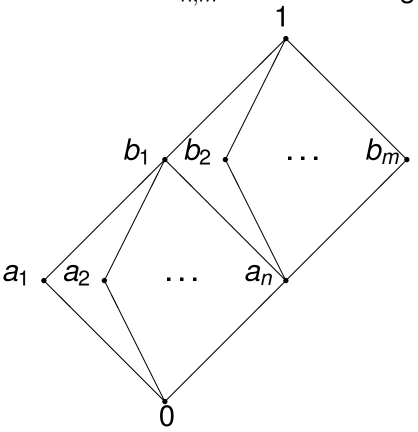

we denote $\mathcal{M}_{n,m}$ the variety generated by $M_{n,m}$

Critical points

Corollary

The following equalities hold

$$\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}) = \aleph_2;$$
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That is all

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This can be extended to the unbounded case.