On positive quasigroups

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Quasigroups

A **quasigroup** is an algebra $(Q, \cdot, \backslash, /)$ with three binary operations satisfying the following identities:

$$y\backslash(y \cdot x) \approx x; \qquad (x \cdot y)/y \approx x$$

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A **loop** is a quasigroup $(Q, \cdot, \backslash, /)$ with an identity element for $(Q, \cdot)$, i.e., an algebra $(Q, \cdot, \backslash, /, 1)$ which satisfies the quasigroup's identities and identities $x \cdot 1 \approx 1 \cdot x \approx x$.

Partially ordered quasigroups

A quasigroup $(Q, \cdot, \backslash, /)$ with a binary relation $\leq$ will be called a **partially ordered quasigroup** if

(i) $(Q, \leq)$ is a partially ordered set;

(ii) for all $x, y, a \in Q$, $x \leq y \Leftrightarrow ax \leq ay \Leftrightarrow xa \leq ya$.

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If $(Q, \cdot, \backslash, /)$ is a loop, then we say that $\mathbf{Q}$ is a partially ordered loop.

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A partially ordered quasigroup $\mathbb{Q}$ is called a **Riesz quasigroup** if it is directed and satisfies the following interpolation property

$$\text{for all } a_i, b_j \in Q \text{ where } a_i \leq b_j,\ i, j \in \{1, 2\},$$

$$\text{there exists } c \in Q \text{ such that } a_i \leq c \leq b_j.$$

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A partially ordered quasigroup $\mathbb{Q}$ is called a **lattice ordered quasigroup** if it has a lattice-order.

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P_Q - the set of all positive elements from Q
N_Q - the set of all negative elements from Q

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- Every partially ordered group $\mathbf{G}$ is a positive quasigroup with $P_G = \{p \in Q : \ p \geq 1\}$
- Every partially ordered loop $\mathbf{Q}$ is a positive quasigroup with $P_Q = \{p \in Q : \ p \geq 1\}$

Positive quasigroups

Positive quasigroup which is not a loop

$$Q = R \times R$$

with the operation

$$(x, y) \cdot (u, v) = (x + u, 2y + v)$$

and the relation $\leq$, where

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The set of all positive elements is $P_Q = \{(x, y) : x > 0\}$.

Positive quasigroups

The concept of a positive quasigroup (and also a left positive and a right positive quasigroup) was introduced by V. M. Tararin (1979). He investigated the linearly ordered positive quasigroups. Further, left positive quasigroups and left positive Riesz quasigroups were studied by V. A. Testov, who characterized them using local left units (1982,1985).

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II. type: $E = \{e, f\}$, e covers f or vice versa

$$e \cdot e = e,\ ef = fe = f$$

III. type: $|E| \geq 2$, E is a trivially ordered set

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Theorem

Let $\mathbf{Q}$ be a positive quasigroup.

(i) $\mathbf{Q}$ is a linearly ordered quasigroup if and only if $Q = P_Q \cup N_Q$;

(ii) $\mathbf{Q}$ is a directed quasigroup if and only if $Q = P_Q \cdot N_Q$;

(iii) $\mathbf{Q}$ is a lattice ordered quasigroup if and only if $\mathbf{Q}$ is directed and P_Q is a lattice.

Congruence relations

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a convex congruence relation on partially ordered quasigroup $\mathbf{Q}$ if there exists $a \in Q$ such that $[a]\theta$ is a convex subset of Q;

a non-trivially ordered congruence relation on partially ordered quasigroup $\mathbf{Q}$ if there exists $a \in Q$ such that $[a]\theta$ is a non-trivially ordered subset of Q.

Congruence relations

Let $\mathbf{Q}$=$(Q, \cdot, \backslash, /, \leq)$ be a partially ordered quasigroup.
Let Q/θ be a quotient-quasigroup of a quasigroup $(Q, \cdot, \backslash, /)$ over its congruence relation θ.
Let us define

$$[x]\theta \leq [y]\theta \text{ iff there exist } a \in [x]\theta, b \in [y]\theta \text{ such that } a \leq b \quad (1)$$

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Let us define

$$[x]\theta \leq [y]\theta \text{ iff there exist } a \in [x]\theta, b \in [y]\theta \text{ such that } a \leq b \quad (1)$$

Theorem

A quotient-quasigroup Q/θ with the relation defined by (1) is a partially ordered quasigroup if and only if θ is a convex congruence relation on $\mathbf{Q}$.

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Theorem

If $\mathbf{Q}$ is a positive quasigroup and θ is a convex non-trivially ordered congruence relation on $\mathbf{Q}$, then $\mathbf{Q}/\theta$ is a partially ordered loop.

Congruence relations

Theorem

The convex directed congruence relations of a Riesz quasigroup $\mathbf{Q}$ form a distributive sublattice in the lattice of all congruence relations of $\mathbf{Q}$.

Ideals

We say that a subquasigroup H is a **normal subquasigroup** of $(Q, \cdot, \backslash, /)$ if there exists a congruence relation θ on $(Q, \cdot, \backslash, /)$ such that $H = [h]\theta$ for some $h \in H$.

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An **o-ideal** of a partially ordered quasigroup $\mathbf{Q}$ is any normal convex directed subquasigroup of $\mathbf{Q}$.

Lemma

If $\mathbf{Q}$ is a positive quasigroup and A is an o-ideal of $\mathbf{Q}$, $|A| > 1$, then $E \subseteq A$.

Ideals and congruences

Theorem

Let $\mathbf{Q}$ be a positive quasigroup and let e be any local unit element from Q. There exists a one-to-one correspondence between the all non-trivially ordered o-ideals A of $\mathbf{Q}$ and all directed convex non-trivially ordered congruence relations on $\mathbf{Q}$. This correspondence is given by

$$\varphi : A \mapsto \theta_A \text{ and } \psi : \theta \mapsto [e]\theta,$$

where $\theta_A = \{(x, y) \in Q \times Q : x/(e\backslash y) \in A\}$.

Ideals and congruences

There exists a one-to-one correspondence between the all o-ideals A of a lattice ordered positive quasigroup $\mathbf{Q}$ and all directed convex congruence relations of $\mathbf{Q}$. This correspondence is given by

$$\varphi : A \mapsto \theta_A \text{ and } \psi : \theta \mapsto [e]\theta,$$

where e is an idempotent element from $\mathbf{Q}$ and

$$\theta_A = \{(x, y) \in Q \times Q : x/(e\backslash y) \in A\}.$$

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Theorem

Let $\mathbf{Q}$ be a positive Riesz quasigroup. Then all non-trivially ordered o-ideals A of $\mathbf{Q}$ form a distributive sublattice in the lattice of all normal subquasigroups of $\mathbf{Q}$ containing E.

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Theorem

The o-ideals of a lattice ordered positive quasigroup $\mathbf{Q}$ form a distributive sublattice in the lattice of all normal subquasigroups of Q.

Homomorphisms

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Definition

Let $\mathbf{Q}$, $\mathbf{H}$ be a partially ordered quasigroups. A map $\varphi : Q \to H$ is said to be an

- **o-homomorphism** if φ is an order-preserving quasigroup homomorphism;
- **o-epimorphism** if φ is a surjective o-homomorphism and there exists $a \in Q$ such that $U_{\varphi(a)} \subseteq \varphi(U_a)$;
- **o-isomorphism** if φ is a one-to-one o-epimorphism.

Homomorphisms

Theorem

A congruence relation θ on partially ordered quasigroup $\mathbf{Q}$ is the kernel of an o-epimorphism if and only if it is convex. If φ is an o-epimorphism of $\mathbf{Q}$ onto $\mathbf{H}$, then $\mathbf{H}$ and $\mathbf{Q}/\text{Ker } \varphi$ are o-isomorphic.

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Corollary

Let φ: $\mathbf{Q} \to \mathbf{H}$ be an o-epimorphism. Let $\mathbf{Q}$ be a positive quasigroup. Then

(i) $\mathbf{H}$ is a positive quasigroup;

(ii) if Ker φ is a non-trivially ordered congruence relation on $\mathbf{Q}$, then $\mathbf{H}$ is a partially ordered loop.