ALGEBRAS WITH SUPPORTS

Jānis Cīrulis
Department of Computer Science
University of Latvia

SSAOS 46: Třešt', September 2008

OVERVIEW

1. Definitions
2. Typical constructions of supported algebras
3. Examples
4. Categories of locally finite algebras

1. DEFINITIONS

1. DEFINITIONS

Initial data:

- $(L, \cap)$ – a meet semilattice

  (sometimes – with the greatest element U),

- $L_0 := \{X \in L\colon (X] \text{ is finite}\}$ – a join-dense ideal in L

i.e., L_0 is a down-set closed under existing finite joins, and every element of L is the join of some subset of L_0.

1. DEFINITIONS

Initial data:

- $(L, \cap)$ – a meet semilattice (sometimes – with the greatest element U),
- $L_0 := \{X \in L\colon (X] \text{ is finite}\}$ – a join-dense ideal in L

i.e., L_0 is a down-set closed under existing finite joins, and every element of L is the join of some subset of L_0.

The motivating standart example:

L – the semilattice of all subsets of some set U (normally infinite),

L_0 – the set of all finite subsets of U.

1. DEFINITIONS

Initial data:

- $(L, \cap)$ – a meet semilattice
  (sometimes – with the greatest element U),
- $L_0 := \{X \in L\colon (X] \text{ is finite}\}$ – a join-dense ideal in L

i.e., L_0 is a down-set closed under existing finite joins, and every element of L is the join of some subset of L_0.

The motivating standart example:

L – the semilattice of all subsets of some set U
(normally infinite),

L_0 – the set of all finite subsets of U.

Elements of L belonging to L_0 will be called *finite*.

1. DEFINITIONS

Initial data:

- $(L, \cap)$ – a meet semilattice

  (sometimes – with the greatest element U),
- $L_0 := \{X \in L\colon (X]$ is finite$\}$ – a join-dense ideal in L

i.e., L_0 is a down-set closed under existing finite joins, and every element of L is the join of some subset of L_0.

The motivating standart example:

L – the semilattice of all subsets of some set U

(normally infinite),

L_0 – the set of all finite subsets of U.

Elements of L belonging to L_0 will be called *finite*.

- A – an arbitrary algebra.

A *support relation*, or *supporting*, *for* A is a relation $\mathbf{spp} \subseteq L \times A$ such that,

- for every $a \in A$, the set $spp(a) := \{X \in L\colon X \;\mathbf{spp}\; a\}$ is an upper set in L, and
- for every $X \in L$, the set $A_X := \{a \in A\colon X \;\mathbf{spp}\; a\}$ is a subalgebra of A.

A *support relation*, or *supporting*, *for* A is a relation $\mathbf{spp} \subseteq L \times A$ such that,

- for every $a \in A$, the set $spp(a) := \{X \in L\colon X \mathbf{\ spp\ } a\}$ is an upper set in L, and
- for every $X \in L$, the set $A_X := \{a \in A\colon X \mathbf{\ spp\ } a\}$ is a subalgebra of A.

Then $(A_X\colon X \in L)$ is a local family of subalgebras, i.e.,

- A_X is a subalgebra of A_Y whenever $X \subseteq Y$, and
- $A = \bigcup(A_X\colon X \in L)$.

(If L has the greatest element U, then $A = A_U$.)

A *support relation*, or *supporting*, *for* A is a relation $\mathbf{spp} \subseteq L \times A$ such that,

- for every $a \in A$, the set $spp(a) := \{X \in L\colon X \ \mathbf{spp}\ a\}$ is an upper set in L, and
- for every $X \in L$, the set $A_X := \{a \in A\colon X \ \mathbf{spp}\ a\}$ is a subalgebra of A.

Then $(A_X\colon X \in L)$ is a local family of subalgebras, i.e.,

- A_X is a subalgebra of A_Y whenever $X \subseteq Y$, and
- $A = \bigcup(A_X\colon X \in L)$.

(If L has the greatest element U, then $A = A_U$.)

The relation $\mathbf{spp}$ is *normal* if

- every set $spp(a)$ is actually a semilattice filter

or, equivalently,

- if the corresponding local family is *multiplicative*:

  $A_X \cap A_Y = A_{X \cap Y}$ for all $X, Y \in L$.

A *supported algebra* is a pair $(A, \mathbf{spp})$, where $\mathbf{spp}$ is a supporting for A.

Elements of $spp(a)$ are said to be *supports of* a.

The subalgebras A_X are called *components* of A.

A *supported algebra* is a pair $(A, \mathbf{spp})$, where $\mathbf{spp}$ is a supporting for A.

Elements of $spp(a)$ are said to be *supports of* a.

The subalgebras A_X are called *components* of A.

A component A_X is *finite-dimensional*, if X is finite.

The union of all such components is a subalgebra of A.

The algebra A is said to be *locally finite-dimensional*, or just *locally finite*, if it coincides with this union.

A *supported algebra* is a pair $(A, \mathbf{spp})$, where $\mathbf{spp}$ is a supporting for A.
Elements of $spp(a)$ are said to be *supports of* a.
The subalgebras A_X are called *components* of A.

A component A_X is *finite-dimensional*, if X is finite.
The union of all such components is a subalgebra of A.
The algebra A is said to be *locally finite-dimensional*, or just *locally finite*, if it coincides with this union.
This is the case if and only if every element of A has a finite support.

More generally:

The signature of A may be related to L and depend on the size of this set in some way. Then a component A_X usually is not similar to A, but may be, for example, a subreduct of A (and of A_Y, if $X \subseteq Y$).

More generally:

The signature of A may be related to L and depend on the size of this set in some way. Then a component A_X usually is not similar to A, but may be, for example, a subreduct of A (and of A_Y, if $X \subseteq Y$).
Then a "natural" support relation for A can frequently be defined in terms of operations of A.

2. TYPICAL CONSTRUCTIONS OF SUPPORT RELATIONS

2. TYPICAL CONSTRUCTIONS OF SUPPORT RELATIONS

Supports by local families:

A – an algebra.

A local family $(A_X\colon X \in L)$ of subalgebras of A

i.e.,

- $A_X \subseteq A_Y$ whenever $X \subseteq Y$, and
- $\bigcup(A_X\colon X \in L) = A$.

2. TYPICAL CONSTRUCTIONS OF SUPPORT RELATIONS

Supports by local families:

A – an algebra.

A local family $(A_X\colon X \in L)$ of subalgebras of A

i.e.,

- $A_X \subseteq A_Y$ whenever $X \subseteq Y$, and
- $\bigcup(A_X\colon X \in L) = A$.

If it is the case, and if $X \ \mathbf{spp}\ a :\equiv a \in A_X$, then $\mathbf{spp}$ is a supporting for A, which is normal iff the family is multiplicative

i.e., if $A_{X \cap Y} = A_X \cap A_Y$.

2. TYPICAL CONSTRUCTIONS OF SUPPORT RELATIONS

Supports by local families:
A – an algebra.
A local family $(A_X \colon X \in L)$ of subalgebras of A
i.e.,
- $A_X \subseteq A_Y$ whenever $X \subseteq Y$, and
- $\bigcup(A_X \colon X \in L) = A$.

If it is the case, and if $X \,\mathbf{spp}\, a :\equiv a \in A_X$, then $\mathbf{spp}$ is a supporting for A, which is normal iff the family is multiplicative
i.e., if $A_{X \cap Y} = A_X \cap A_Y$.

Conversely, components of a supported algebra form a local family. This correspondence between support relations for A and local families in A is one-to-one.

A *finitary* local family in A is an L_0-indexed local family.

A *finitary* local family in A is an L_0-indexed local family.

If $(A_Z\colon Z \in L_0)$ is such a family, and if, for every $X \in L \setminus L_0$,

$$A_X = \bigcup(A_Z\colon Z \in L_0, Z \subseteq X),$$

then

- $(A_X\colon X \in L)$ is a local L-family, and
- the corresponding support relation makes A locally finite.

A *finitary* local family in A is an L_0-indexed local family.

If $(A_Z\colon Z \in L_0)$ is such a family, and if, for every $X \in L \setminus L_0$,

$$A_X = \bigcup(A_Z\colon Z \in L_0, Z \subseteq X),$$

then

- $(A_X\colon X \in L)$ is a local L-family, and
- the corresponding support relation makes A locally finite.

This correspondence between support relations that make A locally finite, and finitary local families in A is also one-to-one.

Supports by independce:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$

ind – relation in $A \times Var$ such that

$$\{a\colon a\ \mathbf{ind}\ x\} \text{ is an subalgebra of } A \text{ for every } x \in Var.$$

(a **ind** x means that a is "independent" of x)

Supports by independce:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$

$\mathbf{ind}$ – relation in $A \times Var$ such that

$$\{a\colon a \ \mathbf{ind}\ x\} \text{ is an subalgebra of } A \text{ for every } x \in Var.$$

(a $\mathbf{ind}$ x means that a is "independent" of x)

If $X \ \mathbf{spp}\ a$ *iff* $x \ \mathbf{ind}\ a$ *for all* $x \notin X$, then $\mathbf{spp}$ is a normal support relation for A.

Supports by independce:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$

$\mathbf{ind}$ – relation in $A \times Var$ such that

$$\{a\colon a \mathbf{\ ind\ } x\} \text{ is an subalgebra of } A \text{ for every } x \in Var.$$

(a $\mathbf{ind}$ x means that a is "independent" of x)

If $X \mathbf{\ spp\ } a$ *iff* $x \mathbf{\ ind\ } a$ *for all* $x \notin X$, then $\mathbf{spp}$ is a normal support relation for A.

Conversely, if $\mathbf{spp}$ is a support relation and

$a \mathbf{\ ind\ } x :\equiv (Var \setminus \{x\}) \mathbf{\ spp\ } a$,

then $\mathbf{ind}$ is an independence relation.

This correspondence *is not* bijective even between *normal* support relations and independence relations.
It *is* if and only if only regular support relations are admitted.

This correspondence *is not* bijective even between *normal* support relations and independence relations.
It *is* if and only if only regular support relations are admitted.

A supporting $\mathbf{spp}$ for A is *regular* if every element of A has the least support.
So, a regular supporting is normal, and every filter $\mathbf{spp}\,(a)$ of L is then principal.

This correspondence *is not* bijective even between *normal* support relations and independence relations.
It *is* if and only if only regular support relations are admitted.

A supporting $\mathbf{spp}$ for A is *regular* if every element of A has the least support.
So, a regular supporting is normal, and every filter $\mathbf{spp}\,(a)$ of L is then principal.

A supporting is regular iff it is induced by an independence relation, which is then unique.

Supports by projectors:

A – an algebra,

$U :=$ the greatest element of L,

P – a homomorphism $L \longrightarrow End(A)$.

Supports by projectors:

A – an algebra,

$U :=$ the greatest element of L,

P – a homomorphism $L \to End(A)$.

$P_X := P(X)$; so

$P_{X \cap Y} = P_X P_Y \quad P_U = \mathrm{id}_A$.

In A: $a \leq b :\equiv a = P_X(b)$ for some X.

Then $\leq$ is a partial order, and every P_X is an interior operator on A (a "projector" onto A_X).

Supports by projectors:

A – an algebra,

$U :=$ the greatest element of L,

P – a homomorphism $L \to End(A)$.

$P_X := P(X)$; so

$P_{X \cap Y} = P_X P_Y \quad P_U = \mathrm{id}_A$.

In A: $a \leq b :\equiv a = P_X(b)$ for some X.
Then $\leq$ is a partial order, and every P_X is an interior operator on A (a “projector” onto A_X).

The family $(\mathrm{ran}\, P_X \colon X \in L)$ is a multiplicative local family in A.
The corresponding supporting is given by X **spp** a *iff* $P_X(a) = a$.

Supports by projectors:
A − an algebra,
$U :=$ the greatest element of L,
P − a homomorphism $L \to End(A)$.

$P_X := P(X)$; so
$P_{X \cap Y} = P_X P_Y \quad P_U = \mathrm{id}_A$.

In A: $a \leq b :\equiv a = P_X(b)$ for some X.
Then $\leq$ is a partial order, and every P_X is an interior operator on A (a "projector" onto A_X).

The family ($\mathrm{ran}\, P_X$: $X \in L$) is a multiplicative local family in A.
The corresponding supporting is given by X **spp** a *iff* $P_X(a) = a$.

It goes also for *algebras with projectors* $(A, P_X)_{X \in L}$.

Supports by substitutions:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$,

$T :=$ the transformation monoid of Var,

$\varepsilon :=$ the neutral element of T,

S – a homomorphism $T \to End(A)$.

Supports by substitutions:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$,

$T :=$ the transformation monoid of Var,

$\varepsilon :=$ the neutral element of T,

S – a homomorphism $T \to End(A)$.

$S_\alpha := S(\alpha)$; so

$$S_{\alpha\beta} = S_\alpha S_\beta, \qquad S_\varepsilon = \mathrm{id}_A.$$

(S_α is the "substitution" over A induced by the transformation α.)

Supports by substitutions:

A – an algebra,

Var – a set of variables,

$L := \mathcal{P}(Var)$,

$T :=$ the transformation monoid of Var,

$\varepsilon :=$ the neutral element of T,

S – a homomorphism $T \to End(A)$.

$S_\alpha := S(\alpha)$; so

$S_{\alpha\beta} = S_\alpha S_\beta, \quad S_\varepsilon = \mathrm{id}_A$.

(S_α is the "substitution" over A induced by the transformation α.)

If X **spp** a is defined by

for all $\alpha, \beta \in T$, $S_\alpha(a) = S_\beta(a)$ *whenever* $\alpha|X = \beta|X$,

then **spp** is a normal supporting for A.

Supports by substitutions:

A – an algebra,
Var – a set of variables,
$L := \mathcal{P}(Var)$,
$T :=$ the transformation monoid of Var,
$\varepsilon :=$ the neutral element of T,
S – a homomorphism $T \to End(A)$.

$S_\alpha := S(\alpha)$; so
$S_{\alpha\beta} = S_\alpha S_\beta$, $S_\varepsilon = \mathrm{id}_A$.
(S_α is the "substitution" over A induced by the transformation α.)

If X **spp** a is defined by
for all $\alpha, \beta \in T$, $S_\alpha(a) = S_\beta(a)$ *whenever* $\alpha|X = \beta|X$,
then **spp** is a normal supporting for A.

It goes* also for *algebras with substitutions* $(A, S_\alpha)_{\alpha \in T}$
(A_X is closed under S_α only if α is constant outside X).

3. EXAMPLES

3. EXAMPLES

First-order language:

the set of variables: Var,

connectives: $\vee, \wedge, \neg$,

a quantifier: $\exists$,

the algebra of formulas: $(F, \vee, \wedge, \neg, \exists x)_{x \in Var}$.

the algebra of quantifier-free formulas: $(F_0, \vee, \wedge, \neg)$.

3. EXAMPLES

First-order language:

the set of variables: Var,

connectives: $\vee, \wedge, \neg$,

a quantifier: $\exists$,

the algebra of formulas: $(F, \vee, \wedge, \neg, \exists x)_{x \in Var}$.

the algebra of quantifier-free formulas: $(F_0, \vee, \wedge, \neg)$.

$Fr(f) :=$ the set of free variables of a formula f,

$L := \mathcal{P}(Var)$.

3. EXAMPLES

First-order language:

the set of variables: Var,

connectives: $\vee, \wedge, \neg$,

a quantifier: $\exists$,

the algebra of formulas: $(F, \vee, \wedge, \neg, \exists x)_{x \in Var}$.

the algebra of quantifier-free formulas: $(F_0, \vee, \wedge, \neg)$.

$Fr(f) :=$ the set of free variables of a formula f,
$L := \mathcal{P}(Var)$.

Independence relation for both F and F_0:
f **ind** $x :\equiv x \notin Fr(f)$.
The corresponding support relation for both F and F_0:
X **spp** $f :\equiv Fr(f) \subseteq X$.

3. EXAMPLES

First-order language:

the set of variables: Var,
connectives: $\vee, \wedge, \neg$,
a quantifier: $\exists$,
the algebra of formulas: $(F, \vee, \wedge, \neg, \exists x)_{x \in Var}$.
the algebra of quantifier-free formulas: $(F_0, \vee, \wedge, \neg)$.

$Fr(f)$:= the set of free variables of a formula f,
$L := \mathcal{P}(Var)$.

Independence relation for both F and F_0:
f **ind** $x :\equiv x \notin Fr(f)$.
The corresponding support relation for both F and F_0:
X **spp** $f :\equiv Fr(f) \subseteq X$.

Both algebras are locally finite.

Cylindric set algebras:

an ordinal α (possibly, infinite),

a set U,

a Boolean algebra $B := \mathcal{P}(U^\alpha)$,

the k-th cylindrification on B ($k < \alpha$): $\mathsf{c}_k\colon B \to B$,

- $\mathsf{c}_k(b) := \{\varphi \in U^\alpha\colon \varphi$ differs at most at k from some $\psi \in b\}$.

Cylindric set algebras:

an ordinal α (possibly, infinite),

a set U,

a Boolean algebra $B := \mathcal{P}(U^\alpha)$,

the k-th cylindrification on B ($k < \alpha$): $\mathsf{c}_k\colon B \to B$,

- $\mathsf{c}_k(b) := \{\varphi \in U^\alpha\colon \varphi$ differs at most at k from some $\psi \in b\}$.

$(B,\mathsf{c}_k)_{k<\alpha}$ is a (full) “diagonal-free” cylindric set algebra.

$L := \mathcal{P}(\alpha)$.

Cylindric set algebras:

an ordinal α (possibly, infinite),

a set U,

a Boolean algebra $B := \mathcal{P}(U^\alpha)$,

the k-th cylindrification on B ($k < \alpha$): $\mathsf{c}_k\colon B \to B$,

- $\mathsf{c}_k(b) := \{\varphi \in U^\alpha\colon \varphi$ differs at most at k from some $\psi \in b\}$.

$(B,\mathsf{c}_k)_{k<\alpha}$ is a (full) “diagonal-free” cylindric set algebra.
$L := \mathcal{P}(\alpha)$.

The independence relation for this algebra defined by
$b \,\mathbf{ind}\, k :\equiv \mathsf{c}_k(b) = b$
gives rise to a regular support relation for it.

Boolean algebras of (infinitary) relations:

$\alpha, U, B := \mathcal{P}(U^\alpha), L := \mathcal{P}(\alpha)$ as above.

Boolean algebras of (infinitary) relations:

$\alpha, U, B := \mathcal{P}(U^\alpha), L := \mathcal{P}(\alpha)$ as above.

A normal but not regular supporting for B:

$X \ \mathbf{spp}\ b \ :\equiv$

for all $\varphi \in U^\alpha$, $\varphi \in b$ whenever $\varphi|X = \psi|X$ for some $\psi \in b$.

Boolean algebras of (infinitary) relations:

$\alpha, U, B := \mathcal{P}(U^\alpha), L := \mathcal{P}(\alpha)$ as above.

A normal but not regular supporting for B:

$X \;\mathbf{spp}\; b \;\; :\equiv$

for all $\varphi \in U^\alpha$, $\varphi \in b$ whenever $\varphi|X = \psi|X$ for some $\psi \in b$.

Hence, it generally differs from that of the preceding example.

Boolean algebras of (infinitary) relations:

$\alpha, U, B := \mathcal{P}(U^\alpha), L := \mathcal{P}(\alpha)$ as above.

A normal but not regular supporting for B:

$X \ \mathbf{spp} \ b \ \ :\equiv$

for all $\varphi \in U^\alpha$, $\varphi \in b$ whenever $\varphi|X = \psi|X$ for some $\psi \in b$.

Hence, it generally differs from that of the preceding example.

If $X = k < \alpha$, then:

a subset of B_X is treated in algebraic first-order logic as a k-ary relation on U,

an operation $B_k \to U$ may be treated as k-ary operation on U.

Relatively free algebras:

a relatively free algebra on free generators Var: W,

the variety generated by W: $\mathcal{V}(W)$.

Relatively free algebras:

a relatively free algebra on free generators Var: W,
the variety generated by W: $\mathcal{V}(W)$.

$L := \mathcal{P}(Var)$.

Relatively free algebras:

a relatively free algebra on free generators Var: W,
the variety generated by W: $\mathcal{V}(W)$.

$L := \mathcal{P}(Var)$.

A normal supporting relation for W:

$X \mathbf{\ spp\ } w :\equiv$
for every $A \in \mathcal{V}(W)$ and $\varphi, \psi \in Hom(W, A)$,
$\varphi|X = \psi|X$ implies $\varphi(w) = \psi(w)$.

Relatively free algebras:

a relatively free algebra on free generators Var: W,
the variety generated by W: $\mathcal{V}(W)$.

$L := \mathcal{P}(Var)$.

A normal supporting relation for W:

$X \mathbf{\ spp\ } w :\equiv$
for every $A \in \mathcal{V}(W)$ and $\varphi, \psi \in Hom(W, A)$,
$\varphi | X = \psi | X$ implies $\varphi(w) = \psi(w)$.

If $X \neq \emptyset$, then is W_X is the subalgebra of W generated by X.

The supported algebra $(W, \mathbf{spp})$ is locally finite.

Relatively free algebras, II:

Var, W, L as above,

substitution of y for x $(x, y \in L)$:

an endomorphism s_y^x of W such that

- $\mathsf{s}_y^x(x) = y$,
- $\mathsf{s}_y^x(z) = z$, if $z \neq x$.

Relatively free algebras, II:

Var, W, L as above,

substitution of y for x $(x, y \in L)$:

an endomorphism s_y^x of W such that

- $\mathsf{s}_y^x(x) = y$,
- $\mathsf{s}_y^x(z) = z$, if $z \neq x$.

Independence relation for W:

$w \textbf{ ind } x :\equiv$ for all $y \in Var$, $\mathsf{s}_y^x(w) = w$.

Relatively free algebras, II:

Var, W, L as above,

substitution of y for x $(x, y \in L)$:

an endomorphism s_y^x of W such that

- $\mathsf{s}_y^x(x) = y$,
- $\mathsf{s}_y^x(z) = z$, if $z \neq x$.

Independence relation for W:

$w \ \mathbf{ind}\ x :\equiv$ for all $y \in Var$, $\mathsf{s}_y^x(w) = w$.

The corresponding support relation coincides with that defined above.

It goes* also for the algebra $(W, \mathsf{s}_y^x)_{x,y \in Var}$

(A_X is not closed under s_y^x if $y \notin X$).

Algebras of partial functions:

an algebra of partial functions $U \to V$: $(F, \cup, \cap, \setminus)$,

Algebras of partial functions:

an algebra of partial functions $U \to V$: $(F, \cup, \cap, \setminus)$,

$L := \mathcal{P}(U)$.

dom f := the domain of $f \in F$,

$f|X$:= the restriction of f to $X \cap \operatorname{dom} f$.

Algebras of partial functions:

an algebra of partial functions $U \to V$: $(F, \cup, \cap, \setminus)$,

$L := \mathcal{P}(U)$.

dom f := the domain of $f \in F$,

$f|X$:= the restriction of f to $X \cap \operatorname{dom} f$.

A family of projectors for F:

P_X: $f \mapsto f|X \quad (X \in L)$.

The corresponding support relation:

$X \ \mathbf{spp} \ f :\equiv \operatorname{dom} f \subseteq X$

is regular.

Function modules:

a ring R,

a set U,

the module of functions R^U.

Function modules:

a ring R,

a set U,

the module of functions R^U.

$L := \mathcal{P}(U)$.

$P_X :=$ the linear operator on R^U defined by

$$P_X(f)(x) := \begin{cases} f(x), & \text{if } x \in X, \\ 0, & \text{if } x \notin X. \end{cases}$$

Function modules:

a ring R,

a set U,

the module of functions R^U.

$L := \mathcal{P}(U)$.

$P_X :=$ the linear operator on R^U defined by

$$P_X(f)(x) := \begin{cases} f(x), & \text{if } x \in X, \\ 0, & \text{if } x \notin X. \end{cases}$$

The family $(P_X\colon x \in L)$ is a family of projectors on R^U.

The corresponding support relation for R^U:

$X \,\mathbf{spp}\, f :\equiv f(x) = 0$ for all $x \notin X$

is regular.

Menger algebras*:

an ordinal α (possibly, infinite),

an α-dimensional Menger algebra W $:= (W, \circ, e_k)_{k \leq \alpha}$, where

- $\circ$ is a $(1 + \alpha)$-ry operation on W ("composition"),
- every e_k is an element of W (the k-th "selector"),
- the following axioms are fulfilled:
  - $w \circ \mathbf{e} = w$,
  - $e_k \circ \mathbf{v} = v_k$,
  - $(w \circ \mathbf{u}) \circ \mathbf{v} = w \circ (\mathbf{u} \circ \mathbf{v})$.

Menger algebras*:

an ordinal α (possibly, infinite),

an α-dimensional Menger algebra W $:= (W, \circ, e_k)_{k \leq \alpha}$, where

- $\circ$ is a $(1 + \alpha)$-ry operation on W ("composition"),
- every e_k is an element of W (the k-th "selector"),
- the following axioms are fulfilled:
  - $w \circ \mathbf{e} = w$,
  - $e_k \circ \mathbf{v} = v_k$,
  - $(w \circ \mathbf{u}) \circ \mathbf{v} = w \circ (\mathbf{u} \circ \mathbf{v})$.

$L := \mathcal{P}(\alpha)$.

The support relation defined by

$X \ \mathbf{spp}\ w :\equiv$ for all $\mathbf{u}, \mathbf{v} \in W^\alpha$, $\quad w \circ \mathbf{u} = w \circ \mathbf{v}$ whenever $\mathbf{u}|X = \mathbf{v}|X$.

The least support of w is the *rank* of w.

Menger algebras*:

an ordinal α (possibly, infinite),

an α-dimensional Menger algebra W $:= (W, \circ, e_k)_{k \leq \alpha}$, where

- $\circ$ is a $(1+\alpha)$-ry operation on W ("composition"),
- every e_k is an element of W (the k-th "selector"),
- the following axioms are fulfilled:
  - $w \circ \mathbf{e} = w$,
  - $e_k \circ \mathbf{v} = v_k$,
  - $(w \circ \mathbf{u}) \circ \mathbf{v} = w \circ (\mathbf{u} \circ \mathbf{v})$.

$L := \mathcal{P}(\alpha)$.

The support relation defined by

$X \,\mathbf{spp}\, w :\equiv$ for all $\mathbf{u}, \mathbf{v} \in W^\alpha$, $\quad w \circ \mathbf{u} = w \circ \mathbf{v}$ whenever $\mathbf{u}|X = \mathbf{v}|X$.

The least support of w is the *rank* of w.

Locally finite Menger algebras of dimension ω are, essentially, abstract finitary clones.

4. CATEGORIES OF LOCALLY FINITE ALGEBRAS

4. CATEGORIES OF LOCALLY FINITE ALGEBRAS

$(A, \mathbf{spp})$, $(A', \mathbf{spp}')$ – supported algebras of the same type.

h is a *homomorphism* $(A, \mathbf{spp}) \to (A', \mathbf{spp}')$ if

- h is a homomorphism $A \to A'$, and
- $X \ \mathbf{spp}\ a$ implies $X \ \mathbf{spp}'\ h(a)$ for all $X \in L$ and $a \in A$.

If, morever, always

- $X \ \mathbf{spp}'\ h(a)$ implies $X \ \mathbf{spp}\ a_0$ for some $a_0 \in h^{-1}(a)$

then the homomorphism h is said to be *full*.

$\mathcal{V}$ – a variety of algebras.

${}_{spp}\mathcal{V}$:= the category of supported algebras from $\mathcal{V}$ and full homomorphisms,

$\mathsf{Lf}_{spp}\mathcal{V}$ – its full subcategory consisting of locally finite algebras.

$\mathcal{V}$ – a variety of algebras.

${}_{spp}\mathcal{V}$:= the category of supported algebras from $\mathcal{V}$ and full homomorphisms,

$\mathsf{Lf}_{spp}\mathcal{V}$ – its full subcategory consisting of locally finite algebras.

Theorem. Suppose that the set L_0 is directed. Then the category $\mathsf{Lf}_{spp}\mathcal{V}$ is equivalent to a quasivariety of L_0-sorted algebras.

Sketch of proof:

Sketch of proof:

L_0 – a directed set.

$\mathcal{V}^{L_0}$:= the class of direct families $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ of algebras from $\mathcal{V}$, where all homomorphisms f_Y^X injective.

Sketch of proof:

L_0 – a directed set.

$\mathcal{V}^{L_0} :=$ the class of direct families $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ of algebras from $\mathcal{V}$, where all homomorphisms f_Y^X injective.

Such *injective* families are L_0-sorted algebras,
$\mathcal{V}^{L_0}$ is a quasivariety.

Sketch of proof:

L_0 – a directed set.
$\mathcal{V}^{L_0}$:= the class of direct families $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ of algebras from $\mathcal{V}$, where all homomorphisms f_Y^X injective.

Such *injective* families are L_0-sorted algebras,
$\mathcal{V}^{L_0}$ is a quasivariety.

A_∞ := the direct limit of $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$,
f_∞^X := the canonic homomorphism $A_X \to A_\infty$,
$A_{\infty X} := f_\infty^X(A_X)$.

$(A_{\infty X}\colon X \in L_0)$ is a finitary local family in A_∞ .

Sketch of proof:

L_0 – a directed set.

$\mathcal{V}^{L_0}$:= the class of direct families $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ of algebras from $\mathcal{V}$, where all homomorphisms f_Y^X injective.

Such *injective* families are L_0-sorted algebras, $\mathcal{V}^{L_0}$ is a quasivariety.

A_∞ := the direct limit of $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$,
f_∞^X := the canonic homomorphism $A_X \to A_\infty$,
$A_{\infty X} := f_\infty^X(A_X)$.

$(A_{\infty X}\colon X \in L_0)$ is a finitary local family in A_∞ .

This passing from an algebra in $\mathcal{V}^{L_0}$ to its direct limit is functorial, i.e., gives rise to a functor $D\colon \mathcal{V}^{L_0} \to \mathsf{Lf}_{spp}\mathcal{V}$.

Conversely, every locally finite supported algebra is (isomorphic to) the direct limit of the family of its finite-dimensional components:

Conversely, every locally finite supported algebra is (isomorphic to) the direct limit of the family of its finite-dimensional components:

$(A, \mathbf{spp})$ – a locally finite supported algebra,
for $X \subseteq Y \in L_0$, $f_Y^X :=$ the embedding of A_X into A_Y.

Then $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ is a direct family, and A is its direct limit with the canonic embeddings $A_X \to A$.

Conversely, every locally finite supported algebra is (isomorphic to) the direct limit of the family of its finite-dimensional components:

$(A, \mathbf{spp})$ – a locally finite supported algebra,
for $X \subseteq Y \in L_0$, $f_Y^X :=$ the embedding of A_X into A_Y.

Then $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ is a direct family, and A is its direct limit with the canonic embeddings $A_X \to A$.

This transformation of locally finite algebras into direct families of their components is functorial, i.e., gives rise to a functor E: $\mathsf{Lf}_{spp}\mathcal{V} \to \mathcal{V}^{L_0}$.

The functors D and E are mutually inverse up to isomorphisms and establish equivalence of $\mathcal{V}^{L_0}$ and $\mathsf{Lf}_{spp}\mathcal{V}$. $\square$

Conversely, every locally finite supported algebra is (isomorphic to) the direct limit of the family of its finite-dimensional components:

$(A, \mathbf{spp})$ – a locally finite supported algebra,
for $X \subseteq Y \in L_0$, $f_Y^X :=$ the embedding of A_X into A_Y.

Then $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ is a direct family, and A is its direct limit with the canonic embeddings $A_X \to A$.

This transformation of locally finite algebras into direct families of their components is functorial, i.e., gives rise to a functor E: $\mathsf{Lf}_{spp}\mathcal{V} \to \mathcal{V}^{L_0}$.

The functors D and E are mutually inverse up to isomorphisms and establish equivalence of $\mathcal{V}^{L_0}$ and $\mathsf{Lf}_{spp}\mathcal{V}$. $\square$

$(L, L_0$ fixed) Study of locally finite algebras is equivalent to study of algebras with a selected finitary local family of subalgebras.

In general, the support relation for A_∞ induced by a direct family $(A_X, f_Y^X)_{X \subset Y \in L_0}$ is normal iff,

for all $X, Y \subset Z \in L_0$ and $a_1 \in A_X$, $a_2 \in A_Y$,

$f_Z^X(a_1) = f_Z^Y(a_2)$

implies

$a_1 = f_X^{X \cap Y}(b)$ and $a_2 = f_Y^{X \cap Y}(b)$ for some $b \in A_{X \cap Y}$.

Normal support relations defined by projectors:

Normal support relations defined by projectors:

L has the greatest element U,
$\mathcal{V}$ – a variety of algebras.

${}_P\mathcal{V} :=$ the category of algebras $(A, P_X)_{X \in L}$ such that
- $A \in \mathcal{V}$,
- $P_{X \cap Y} = P_X P_Y$ for all $X, Y \in L$,
- $P_U = \mathrm{id}_A$.

$\mathrm{Lf}_P\mathcal{V}$ – its full subcategory consisting of locally finite algebras.

Normal support relations defined by projectors:

L has the greatest element U,
$\mathcal{V}$ – a variety of algebras.

${}_P\mathcal{V}$:= the category of algebras $(A, P_X)_{X \in L}$ such that

- $A \in \mathcal{V}$,
- $P_{X \cap Y} = P_X P_Y$ for all $X, Y \in L$,
- $P_U = \mathrm{id}_A$.

$\mathrm{Lf}_P\mathcal{V}$ – its full subcategory consisting of locally finite algebras.

Theorem. Suppose that the set L_0 is directed. Then the category $\mathrm{Lf}_p\mathcal{V}$ is equivalent to a variety of L_0-sorted algebras.

The variety consists of algebras $(A_X, f_Y^X, g_X^Y)_{X \subseteq Y \in L_0}$, where

- $A_X \in \mathcal{V}$ for all $X \in L_0$,
- $(A_X, f_Y^X)_{X \subseteq Y \in L_0}$ is a direct family,
- $(A_Y, g_X^Y)_{X \subseteq Y \in L_0}$ is an inverse family,
- the mappings g_X^Y: $A_Y \to A_X$ satisfy also the condition

$g_X^Y f_Y^X = \mathrm{id}_{A_X}$,

  - $f_X^{X \cap Y} g_{X \cap Y}^Y = g_X^Z f_Z^Y$ whenever $X, Y \subseteq Z$.

In particular, always $g_X^Y f_Y^X = \mathrm{id}_{A_X}$.