Decomposition of mappings on finite sets.

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ABSTRACT

We are going to present a single mapping that enables to define every mappings on finite sets.

This mapping just uses the elementary arithmetical bricks :

- **the constant 0**
- **the successor function**
- **the equality predicate**
- ~~**the natural order predicate**~~

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[S.B. : Journal of Symbolic Computation, number 37 (2004)]

$$\# : \mathbb{N}^2 \longrightarrow \mathbb{N}$$

$$x \# y = \begin{cases} x+1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

THEOREM

$\forall$S finite subset of $\mathbb{N}$
$\forall$k integer > 0

Every F : $S^k \longrightarrow \mathbb{N}$
is a finite composition
of the commutative operation #

This result is effective

In **Z/10Z**, one obtains decompositions of

	addition	,	product	
with	**426247**		**357982**	symbols #

a) Reduction to Relations : $Im(F) \subseteq \{0,1\}$

Assume that Im(F) contains some integer >1.

Let

$$F_0(X) = \begin{cases} F(X)-1 & \text{if } F(X)>0 \\ 0 & \text{if } F(X)=0 \end{cases}$$

$$F_1(X) = \begin{cases} F(X)-1 & \text{if } F(X)>0 \\ 1 & \text{if } F(X)=0 \end{cases}$$

By construction : $F = F_0 \# F_1$

By induction, it is sufficient to prove the result for mappings $F : S^k \rightarrow \{0,1\}$

b) Constants

Every integer m is definable :

$[0] = x \# (x \# x) \qquad = x \# (x+1)$

and for the expression [m] of m :

$[m+1] = [m] \# [m]$

c) Test of egality

For every integer i, the function

$$IS(x,i) = \begin{cases} 1 \text{ if } x=i \\ 0 \text{ if } x \neq i \end{cases}$$

is definable :

$IS(x,i) = [0] \# ([0] \# (x \# [i]))$

d) Logical OR function

The function OR

OR(0,0)=0 OR(0,1)=1
OR(1,0)=1 OR(1,1)=1

is definable :

OR(x,y) = [0] # ([1] # (x # y))

e) Logical AND function

The function AND

AND(0,0)=0 AND(0,1)=0
AND(1,0)=0 AND(1,1)=1

is definable :

AND(x,y) = ([0] # x) # ([1] # (y # y))

f) Final decomposition

Assume that $Im(F) \subseteq \{0,1\}$.

Let $U=F^{-1}(1)$. Induction on $card(U)$:

- **If $U=\varnothing$ then $F= x_1 \# (x_1 \# x_1)$ (i.e. [0])**

- **If $I=(i_1,i_2,..,i_k) \in U$ then define for $X=(x_1,x_2,..,x_k)$:**

$$F_I(X)= \begin{cases} 0 & \text{if } X=I \\ F(X) & \text{if } X\neq I \end{cases}$$

$$G_I = AND(IS(x_1, i_1) , IS(x_2, i_2) , \ldots, IS(x_k, i_k))$$

$$G_I(X)= \begin{cases} 1 & \text{if } X=I \\ 0 & \text{if } X\neq I \end{cases}$$

$$F = OR (F_I , G_I)$$

What about infinite mappings ?

Negative Result

The mapping $F : \mathbb{N} \rightarrow \mathbb{N}$

$$F(x)=2x$$

is NOT a finite composition of the operation #

Proof :

a term T on # and x of height h can only represent a mapping F where $F(x) \leq x+h$ since
$A \# B \leq \max(A,B)+1$

POSITIVE RESULT

$\forall$ enumerable set $T=\{ t_0 , t_1 , t_2 ,\dots \}$
$\forall k$ integer > 0
$\forall S$ finite subset of T^k

Every $F : S \rightarrow T$
is a finite composition
of the commutative operation

$\# : T^2 \rightarrow T$

$$x \# y = \begin{cases} t_{i+1} & \text{if } x = y = t_i \\ t_0 & \text{if } x \neq y \end{cases}$$

THANKS !