More clones from ideals

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Outline

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Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, and we let $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete Lattice: $\mathbf{CLONE}(X)$.

Definition: For any $C \subseteq \mathcal{O}$ let $\langle C \rangle$ be the clone generated by C.
We write $C(f)$ for $\langle C \cup \{f\} \rangle$.

Size of **CLONE**(X)

If X is finite, then $\mathcal{O}_X$ is countable.

- If $|X| = 1$, then $\mathcal{O}_X$ is trivial.
- If $|X| = 2$, then **CLONE**(X) is countable, and completely understood. ("Post's Lattice")
- If $3 \leq |X| < \aleph_0$, then $|\mathbf{CLONE}(X)| = 2^{\aleph_0}$, and not well understood.

If X is infinite, then

- $|\mathcal{O}_X| = 2^{|X|}$,
- $|\mathbf{CLONE}(X)| = 2^{2^{|X|}}$,
- and only little is known about the structure of **CLONE**(X).

Completeness

Example

The functions $\wedge, \vee, \mathtt{true}, \mathtt{false}$ *do not generate all operations on* $\{\mathtt{true}, \mathtt{false}\}$*.*

Proof: All these functions are monotone, and $\neg$ is not.

Now let X be any set.

Example

Assume that $\leq$ *is a nontrivial partial order on* X*, and that all functions in* $C \subseteq \mathcal{O}$ *are monotone with respect to* $\leq$*.*
Then $\langle C \rangle \neq \mathcal{O}$*.*

Polymorphisms

Let X be a set, $C \subseteq \mathcal{O}_X$.

- If all functions in C respect some order $\leq$ on X,
- or: if all functions in C respect some nontrivial equivalence relation θ
- or: if all functions in C respect some nontrivial fixed set $A \subset X$
  (i.e., $f[A^k] \subseteq A$)
- or ...

then $\langle C \rangle \neq \mathcal{O}$.

We write Pol($\leq$), Pol(θ), Pol(A), ... for the clone of all functions respecting $\leq$, θ, A, ...
Instead of unary (A) or binary ($\leq$, θ) relations, we may also consider n-ary or even infinitary relations.

Outline

Pol( ) and precomplete clones

- Every set of the form $\mathrm{Pol}(A_1) \cap \mathrm{Pol}(A_2) \cap \mathrm{Pol}(\theta_3) \cap \cdots$ is a clone.
- Conversely, every clone is the intersection of sets of the form $\mathrm{Pol}(R)$ (where the R's can be chosen of finite arity if X is finite).

The "maximal" or "precomplete" clones are the coatoms in the clone lattice.

$C \neq \mathcal{O}$ is precomplete iff $C(f) = \mathcal{O}$ for all $f \in \mathcal{O} \setminus C$.

- Every precomplete clone is of the form $\mathrm{Pol}(R)$ for some relation R.

Question

Which relations R give rise to precomplete clones?

This is nontrivial, already for binary relations.

Precomplete clones on finite sets

Example

Let $\emptyset \neq A \neq X$.
Then $\mathrm{Pol}(A)$ *is precomplete.*

Example

Let X *be finite. Let* θ *be a nontrivial equivalence relation.*
Then $\mathrm{Pol}(\theta)$ *is precomplete.*

Theorem (Rosenberg, 1970)

There is an explicit list $(R_i : i \in I)$ *of finitely many (depending on the cardinality of* X*) relations such that* $(\mathrm{Pol}(R_i) : i \in I)$ *lists all precomplete clones on* X.

Moreover, every clone $C \neq \mathcal{O}$ is below some precomplete clone.

Precomplete clones on infinite sets

Example

Let $\emptyset \neq A \neq X$.
Then $\mathrm{Pol}(A)$ is precomplete.

Example

Let θ be a nontrivial equivalence relation **with finitely many classes***.*
Then $\mathrm{Pol}(\theta)$ is precomplete.

- For which R is $\mathrm{Pol}(R)$ precomplete?
- Is every $C \neq \mathcal{O}$ below some precomplete clone?

Outline

Fixpoint clones

Definition

Let $A \subseteq X$. $\mathrm{fix}(A)$ is the set of all functions f satisfying

$\forall x \in A: \ f(x, \ldots, x) = x$.

This is a clone.

Definition

Let F be a filter on X. $\mathrm{fix}((F))$ is defined as $\bigcup_{A \in F} \mathrm{fix}(A)$, i.e.,

$\mathrm{fix}((F)) = \{g : \exists A \in F \, \forall x \in A: \ g(x, \ldots, x) = x\}$.

- $\mathrm{fix}((F))$ is a clone.
- If F is the principal filter generated by the set A, then $\mathrm{fix}((F)) = \mathrm{fix}(A)$.
- larger filter $\Rightarrow$ larger clone.
- maximal filter $\Rightarrow$ maximal clone.

Fixpoint clones, application

Let $C_0 := \mathrm{fix}(X)$, i.e. the clone of all itempotent functions, i.e., functions f satisfying $f(x, \ldots, x) = x$ for *all* $x \in X$.
Let $C_1 := \mathrm{fix}(\emptyset) = \mathcal{O}$, the clone of all functions. Then the interval $[C_0, C_1]$ in the clone lattice is rather complicated, and yet we can "explicitly" describe it.

Theorem (Goldstern-Shelah, 2004)

The clones in the interval $[C_0, C_1]$ are exactly the clones $\mathrm{fix}(\!(F)\!)$, for all possible filters (including the trivial filter $\mathfrak{P}(X)$). (Maximal=precomplete clones correspond to ultrafilters.)

So this interval is order isomorphic to the lattice of closed subsets of βX (with reverse inclusion).

Outline

Clones from ideals

Definition

Let I be a nontrivial ideal on the set X containing all small sets.
$f : X^k \to X$ preserves I if $\forall A \in I : f[A^k] \in I$.
We write $\mathrm{Pol}(\!(I)\!)$ for the set of all functions preserving I.

- $\mathrm{Pol}(\!(I)\!)$ is a clone.
- If I is the principal ideal generated by the set A, then $\mathrm{Pol}(\!(I)\!) = \mathrm{Pol}(A)$.
- larger ideal $\not\Rightarrow$ larger clone.
- maximal ideal $\Rightarrow$ maximal clone.
- However, many other ideals also yield maximal clones.
  $I^{-\circ} := \{A \subseteq X : \forall B \in [A]^\omega : [B]^\omega \cap I \neq \emptyset\}$.
  If $I = I^{-\circ}$, then $\mathrm{Pol}(\!(I)\!)$ is maximal.

Ideal clones, application

For every subset $A \subseteq 2^\omega$ we can find (explicitly) an ideal I_A, such that $I_A = I_A^{-\circ}$, and that the ideals I_A are all different.

Theorem (Beiglböck-Goldstern-Heindorf-Pinsker, 2007)

While the ideals I_A are not maximal, the clones $\mathrm{Pol}(\langle\!\langle I_A \rangle\!\rangle)$ *are (for nontrivial A).*

This gives an explicit example of $2^{\mathfrak{c}}$ many precomplete clones on a countable set. (Even without AC.)

Question

Find such examples on uncountable sets.

Equivalence relations

Example

Let θ be a nontrivial equivalence relation on a finite set. Then $\mathrm{Pol}(\theta)$ is a precomplete clone.

Example

Let θ be a nontrivial equivalence relation on any set, with finitely many classes. Then $\mathrm{Pol}(\theta)$ is a precomplete clone.

Definition

Let $\mathcal{E}$ be a directed family of equivalence relations (coarser and coarser).
Define $\mathrm{Pol}(\!(\mathcal{E})\!)$ *as the set of all functions $f : X^k \to X$ with: for all $E \in \mathcal{E}$ there is $E' \in \mathcal{E}$ such that: whenever $\vec{x}\, E\, \vec{y}$, then $f(\vec{x}) E' f(\vec{y})$.*

When is $\mathrm{Pol}(\!(\mathcal{E})\!)$ precomplete? Difficult. Because…

Fact

For every ideal I there is a family $\mathcal{E}$ as above such that $\mathrm{Pol}(\!(I)\!) = \mathrm{Pol}(\!(\mathcal{E})\!)$.

Outline

Growth clones

Definition
Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ *for simplicity. For every infinite* $A = \{a_0 < a_1 < \cdots\} \subseteq X$ *we define* $\mathrm{bound}(A)$ *as the set of functions which do not jump to far in A:*

$$\mathrm{bound}(A) := \{f : \exists k \forall i : \vec{x} < a_i \Rightarrow f(\vec{x}) < a_{i+k}\}$$

(This is a clone.)

A similar construction is possible for uncountable sets.

Growth clones, continued

Definition

Let $X = \mathbb{N}$ again. For every filter F of subsets of X we define $\mathrm{bound}(\!(F)\!) := \bigcup_{A \in F} \mathrm{bound}(A)$.

$$\mathrm{bound}(\!(F)\!) := \{f : \exists A \in F\, \exists k\, \forall i : \vec{x} < a_i^A \Rightarrow f(\vec{x}) < a_{i+k}^A\}$$

(where $a_0^A < a_1^A < \cdots$ is the increasing enumeration of A).

- $\mathrm{bound}(\!(F)\!)$ is a clone.
- If F is the principal filter generated by the set A, then $\mathrm{bound}(\!(F)\!) = \mathrm{bound}(A)$.
- larger filter $\Longrightarrow$ larger clone.
- maximal filter $\not\Longrightarrow$ maximal clone.
- (In fact, $\mathrm{bound}(\!(F)\!)$ is never a maximal clone.)

Growth clones, application

Theorem (G*-Shelah, 2002)

Assume CH. Then on there is a filter F on $\mathbb{N} = \{0, 1, 2, \dots\}$ such that, letting $C := \mathrm{bound}(\!(F)\!)$, we know the interval $[C, \mathcal{O})$ quite well: it is (more or less) a quite saturated linear order **L** *with no last element.*

(In particular: not every clone is below a precomplete clone.)

We can choose $\mathrm{bound}(\!(F)\!)$ in such a way that the relation $f \leq g \;\Leftrightarrow\; f \in C(g)$ is a linear quasiorder. The clones above C will then be the Dedekind cuts in this order.

This relation $f \leq g$ means that on a large set (i.e., a set in the filter F), g grows at least as fast as f.

Growth clones, new application

Theorem? (Aug-Sep 2008)

Let $\mathbb{N} = N_1 \dot\cup N_2$, with two infinite disjoint sets N_1, N_2, say odd and even numbers.
Assume CH. Then there are filters F_1, F_2 on N_1 and N_2, respectively, such that, letting $C := \mathrm{bound}((F_1)) \cap \mathrm{bound}((F_2))$, we know the interval $[C, \mathcal{O})$ quite well: it is (more or less) $\mathbf{L} \times \mathbf{L}$, with $\mathbf{L}$ the quite saturated linear order from the previous slide.

Theorem? (2009?)

Let $(F_i : i \in I)$ be a family of many (almost?) disjoint sets.
Assume CH. Then there filters F_i, $i \in I$, such that, letting $C := \bigcap_{i \in I} \mathrm{bound}((F_i))$, the interval $[C, \mathcal{O})$ is (more or less) $\mathbf{L}^{\mathbf{I}}$.