(A) $T(x_n) = \left(\frac{1}{n}(x_1+\dots+x_n)\right)_{n=1}^{\infty}$, $T: \ell_1 \to c_0$

a) T zjevně lineární, $x \in \ell_1$, pak $0 \le \lim_{n\to\infty} |(Tx)_n| =$

$= \lim_{n\to\infty} \left|\frac{1}{n}(x_1+\dots+x_n)\right| \le \lim_{n\to\infty} \frac{1}{n}(|x_1|+\dots+|x_n|) \le$

$\le \lim_{n\to\infty} \frac{1}{n}\|x\|_{\ell_1} = 0$, tedy $Tx \in c_0$. Navíc

$\sup_n |(Tx)_n| \le \sup_n \frac{1}{n}\|x\|_1 \le \|x\|_1 \Rightarrow \|T\| \le 1 \Rightarrow T \in \mathcal{L}(X,Y)$

b) $T^*: \ell_1 \to \ell_\infty$, pro $y \in \ell_1$ platí $T^* y = z \in \ell_\infty$ a

$(T^*y)(x) = z(x) = \sum_{k=1}^{\infty} x_k z_k$

$\| $

$y(Tx) = \sum_{n=1}^{\infty} (Tx)_n y_n = \sum_{n=1}^{\infty} \frac{1}{n}(x_1+\dots+x_n) y_n =$

$= \sum_{1 \le n \le \infty} \sum_{1 \le k \le n} \frac{1}{n} y_n x_k = \sum_{k=1}^{\infty} x_k \left(\sum_{n=k}^{\infty} \frac{1}{n} y_n\right)$, $x \in c_0$

$\Big\lceil$ F.V.: $\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{n}|y_n||x_k| \le \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{n}|y_n|\,\|x\|_\infty = \sum_{n=1}^{\infty}|y_n|\,\|x\|_\infty =$

$= \|x\|_\infty \|y\|_1 < \infty \Big\rfloor$

dosadíme $x = e_k$ (bázové vektory), pak dostaneme

$z_k = \sum_{n=k}^{\infty} \frac{1}{n} y_n$, $k \in \mathbb{N}$. Tedy

$(T^*y)_k = \sum_{n=k}^{\infty} \frac{1}{n} y_n$, $k \in \mathbb{N}$.

c) Uvažujme e_k bázové vektory, pak $\|Te_k\|_\infty = \|(0,\dots,0,\frac{1}{k},\frac{1}{k+1},\frac{1}{k+2},\dots)\|_\infty$

$= \frac{1}{k} \to \infty$.

Tedy neexistuje $c > 0$, že $\frac{1}{k} = \|Te_k\|_\infty \ge c\|e_k\|_{\ell_1} = c > 0$, $k \in \mathbb{N}$.

Proto T není izomorfizmus do.

d) $T: c_0 \to c_0$ je třeba ukázat. Ať $x \in c_0$, $\varepsilon > 0$.

Nechť $n_0 \in \mathbb{N}$, že $|x_n| < \frac{\varepsilon}{2}$ pro $n \geq n_0$. Ať $m_0 > n_0$, že

$\frac{n_0 \|x\|_\infty}{m_0} < \frac{\varepsilon}{2}$. Pak pro $m > m_0$ platí

$$\left| \frac{x_1 + \dots + x_{n_0} + \dots + x_m}{m} \right| \leq \frac{|x_1 + \dots + x_{n_0}|}{m} + \frac{|x_{n_0+1} + \dots + x_m|}{m} \leq$$

$$\leq \frac{n_0 \|x\|_\infty}{m} + \frac{(m - n_0)}{m} \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow T \text{ je do } c_0.$$

Zjevně T lineární a $\|Tx\|_\infty \leq \sup_n \frac{1}{n}(|x_1| + \dots + |x_n|) \leq$

$$\leq \sup_n \frac{1}{n}(n \|x\|_\infty) \leq \|x\|_\infty$$

$\Rightarrow \|T\| \leq 1$.

(B) a) pro $n \in \mathbb{N}$ ať $f_n = \chi_{(\frac{1}{n+1}, \frac{1}{n})}$. Pak

$\{f_n : n \in \mathbb{N}\}$ je LN množina v $L_3([0,\pi])$, neboť

$0 = \sum_{n=1}^{k} a_n f_n \Rightarrow a_1 = a_2 = \dots = a_k = 0$, neboť f_n mají disjunktní nosiče.

b) $\sin = f_1$ $\quad \varphi_1(f) = \int_0^\pi f \cos$ $\quad$ jelikož $\cos, \sin \in L_{3'}([0,\pi])$, tedy

$\cos = f_2$ $\quad \varphi_2(f) = \int_0^\pi f \sin$ $\qquad \frac{1}{3} + \frac{1}{3'} = 1$

$$\varphi_1, \varphi_2 \in X^*, \; f_1, f_2 \in X \Rightarrow Tf = \sum_{i=1}^{2} \varphi_i(f) f_i \in F(X) \subset L(X).$$

c) $F(X) \subset K(X) \Rightarrow T$ kompaktní

d) • $\lambda = 0 \Rightarrow \operatorname{Ker} \varphi_1 \cap \operatorname{Ker} \varphi_2$ má kodimenzi nejvýše 2, $\dim X = \infty$, tedy

$\dim(\operatorname{Ker} \varphi_1 \cap \operatorname{Ker} \varphi_2) = \infty \Rightarrow \operatorname{Ker} T \neq \{0\} \Rightarrow \lambda \in \sigma_p(T)$

• $\lambda \neq 0 \Rightarrow \lambda f = \sum_{i=1}^{2} \varphi_i(f) f_i \Rightarrow f = \sum_{i=1}^{2} a_i f_i \Rightarrow$

$$\Rightarrow \sum_{i=1}^{2} \lambda a_i f_i = \sum_{i=1}^{2} f_i \Big(\varphi_i \big(\sum_{j=1}^{2} a_j f_j \big) \Big) = \sum_{i=1}^{2} f_i \Big(\sum_{j=1}^{2} a_j \varphi_i(f_j) \Big) =$$

$$\Big[\; \varphi_1(f_1) = \int_0^\pi \sin \cos = \int_0^\pi \tfrac{1}{2} \sin 2t = -\tfrac{1}{2} \Big[\tfrac{\cos 2t}{2} \Big]_0^\pi = -\tfrac{1}{4}(1-1) = 0$$

$$\varphi_1(f_2) = \int_0^\pi \cos^2 t = \int_0^\pi \frac{1+\cos 2t}{2} = \frac{\pi}{2} + \frac{1}{2} \Big[\frac{\sin 2t}{2} \Big]_0^\pi = \frac{\pi}{2}$$

$$\varphi_2(f_1) = \int_0^\pi \sin^2 t = \int_0^\pi 1 - \cos^2 t = \pi - \frac{\pi}{2} = \pi/2$$

$$\varphi_2(f_2) = \int_0^\pi \cos \sin = 0 \; \Big]$$

$$= f_1 \Big(a_2 \frac{\pi}{2} \Big) + f_2 \Big(a_1 \frac{\pi}{2} \Big)$$

$\Rightarrow \lambda a_1 f_1 + \lambda a_2 f_2 = f_1 \big(a_2 \frac{\pi}{2} \big) + f_2 \big(a_1 \frac{\pi}{2} \big)$, $\; f_1, f_2 \in LN \Rightarrow$

$\Rightarrow \quad \lambda a_1 = a_2 \frac{\pi}{2} \quad \Rightarrow a_1 = a_2 \frac{\pi}{2} \cdot \frac{1}{\lambda} \Rightarrow \lambda a_2 = \left(\frac{\pi}{2}\right)^2 a_2 \cdot \frac{1}{\lambda}$

$\qquad \lambda a_2 = a_1 \frac{\pi}{2} \qquad\qquad (\lambda^2 - (\frac{\pi}{2})^2) a_2 = 0$

$\Rightarrow a_2 = 0 \Rightarrow a_1 = 0 \Rightarrow f = 0$

ii) $a_2 \neq 0 \Rightarrow \lambda = \pm \frac{\pi}{2}$

- $\lambda = \frac{\pi}{2} \Rightarrow a_1 = a_2 \Rightarrow a_1 = a_2 = 1 \Rightarrow \sin + \cos$ vl. vektor pro $\frac{\pi}{2}$

- $\lambda = -\frac{\pi}{2} \Rightarrow -a_1 = a_2 \Rightarrow \begin{matrix} a_1 = 1 \\ a_2 = -1 \end{matrix} \Rightarrow \sin - \cos$ vl. vektor pro $-\frac{\pi}{2}$

$\Rightarrow \sigma_p(T) = \{0, \frac{\pi}{2}, -\frac{\pi}{2}\}$.

c) T kpt a $\Rightarrow \sigma(T) = \{0\} \cup \sigma_p(T) = \{0, \frac{\pi}{2}, -\frac{\pi}{2}\}$.

$\dim X = \infty$