Příklad A:

a) $e_1 = f_1 = 1$, $e_2 = e_1 + a f_2 = 1 + ax^2$ s podmínkou

$$0 = \langle e_2, 1\rangle = \int_{-1}^{1} (1+ax^2)\,1 = 2 + a\frac{2}{3} \Rightarrow a = -3$$

$$\Rightarrow e_1 = 1,\ e_2 = 1-3x^2$$

$$\|e_1\|^2 = \int_{-1}^{1} 1^2 = 2 \Rightarrow \|e_1\| = \sqrt{2}$$

$$\|e_2\|^2 = \int_{-1}^{1} (1-3x^2)^2 = 2\int_0^1 (1+9x^4-6x^2) = \frac{8}{5} \Rightarrow \|e_2\| = \sqrt{\frac{8}{5}}$$

$$\Rightarrow \Big\{ \underbrace{\frac{1}{\sqrt{2}}}_{\tilde e_1},\ \underbrace{\frac{1-3x^2}{\sqrt{\frac{8}{5}}}}_{\tilde e_2} \Big\} \text{ je ON-báze } Y$$

b) $f_3(x) = x$ je v $Y^\perp \Rightarrow P_Y f_3 = \langle f_3, \tilde e_1\rangle \tilde e_1 + \langle f_3, \tilde e_2\rangle \tilde e_2 = 0$

$\Rightarrow g \equiv 0$ a $\operatorname{dist}(f_3, Y) = \|f_3 - P_Y f_3\| = \|f_3\| = \sqrt{\int_{-1}^{1} x^2} = \sqrt{\frac{2}{3}}$

c) $P: Z \to Z$ projekce, $0 \neq P \neq I \Rightarrow Z = \operatorname{Ker} P \oplus_t \operatorname{Rng} P$, přičemž $\{0\} \neq \operatorname{Ker} P \neq Z \neq \operatorname{Rng} P$. Tedy ex. $x_{Ker} \in \operatorname{Ker} P \setminus \{0\}$, $x_{Rng} \in \operatorname{Rng} P \setminus \{0\}$. Pak

- $P x_{Ker} = 0 \cdot x_{Ker} \Rightarrow x_{Ker}$ je vlastní vektor pro 0
- $P x_{Rng} = 1 \cdot x_{Rng} \Rightarrow x_{Rng}$ — // — pro 1

$\Rightarrow \{0,1\} \subset \sigma_p(P)$.

- $\lambda \notin \{0,1\}$: $(\lambda I - P)x = y$

$$\lambda x - Px = y$$
$$\lambda x = y + Px$$
$$\lambda P x = Py + Px$$
$$Px(\lambda - 1) = Py$$
$$Px = \frac{Py}{\lambda - 1}$$

$\Rightarrow \lambda x = y + \frac{Py}{\lambda-1}$

$x = \frac{y}{\lambda} + \frac{Py}{\lambda(\lambda-1)}$, tj.

$y \mapsto \frac{y}{\lambda} + \frac{Py}{\lambda(\lambda-1)}$ je spojitá inverze k $(\lambda I - P)^{-1}$. Proto $\sigma_p(P) = \sigma(P) = \{0,1\}$.

Příklad 3: $X = L_2([0,1])$

a) Ať $f_n = \chi_{(1/n+1, 1/n)}$, $n \in \mathbb{N}$. Pak $\{f_n : n \in \mathbb{N}\} \subset X$ je LN množina:

$$0 = \sum_{i=1}^{n} a_i f_i \Rightarrow a_1 = a_2 = \dots = a_n = 0, \text{ neboť } f_i \text{ mají disjunktní nosiče}$$

b) $f \in X \Rightarrow t f(t) \in L_1([0,1]) \Rightarrow Tf \in AC[0,1] \Rightarrow Tf \in X$.

$$\|Tf\|_{L_2}^2 = \int_0^1 \left| \int_0^x t f(t)\, dt \right|^2 dx \leq \int_0^1 \left(\int_0^1 |f(t)|\, dt \right)^2 dx =$$

$$= \left(\int_0^1 |f| \right)^2 \overset{\text{Hölder}}{\leq} \left[\left(\int_0^1 |f|^2 \right)^{1/2} \left(\int_0^1 1^2 \right)^{1/2} \right]^2 \leq \left(\int_0^1 |f|^2 \right)^1$$

$$\Rightarrow \|Tf\|_X \leq \|f\|_X \Rightarrow \|T\| \leq 1$$

- zřejmě T lineární

c) $T B_X \subset C[0,1]$, přičemž: $\cdot\, f \in B_X \Rightarrow \|Tf\|_\infty \leq \int_0^1 |f(t)|\, dt \leq$

$$\leq \left(\int_0^1 |f(t)|^2 \right)^{1/2} \cdot 1 \leq 1$$

$\cdot\, f \in B_X$, $0 \leq x < y \leq 1 \Rightarrow |Tf(y) - Tf(x)| = \left| \int_x^y t f(t)\, dt \right| \leq$

$$\leq \int_x^y t |f(t)| \leq \left(\int_x^y t^2 \right)^{1/2} \underbrace{\left(\int_x^y |f|^2 \right)^{1/2}}_{\leq 1} \leq \sqrt{\tfrac{1}{3}(y^3 - x^3)} \cdot 1 \leq$$

$$\leq \sqrt{\tfrac{1}{3}(y-x)\underbrace{(y^2 + yx + x^2)}_{\leq 3}} \leq \sqrt{y-x}$$

Tedy pro $\varepsilon > 0$ ex. $\delta > 0$, že pro $|y-x| < \delta$ a $f \in B_X$ je $|Tf(y) - Tf(x)| < \varepsilon$

$\Rightarrow$ TB_X splňuje předpoklady Arzela-Ascoli $\Rightarrow$

$\Rightarrow$ TB_X je rel. kpt. v $C[0,1]$.

Ať $\{f_n\} \subset B_X$ daná, pak ex. $\{n_k\}$ rostoucí, že $\{Tf_{n_k}\}$ je stejnoměrně cauchyovská. Pak $\|Tf_{n_k} - Tf_{n_l}\|_X^2 = \int_0^1 |Tf_{n_k} - Tf_{n_l}|^2 \leq$

$$\leq \|Tf_{n_k} - Tf_{n_l}\|_\infty^2 \Rightarrow \{Tf_{n_k}\} \text{ je cauchy v } X$$

$\Rightarrow$ T je kptní

d). $\lambda = 0$: $Tf = 0 \Rightarrow t f(t) \in L_1[0,1] \Rightarrow Tf \in AC[0,1]$ a $(Tf)' = t f(t)$ s.v.

Proto $0 = Tf \Rightarrow 0 = (Tf)' = t f(t)$ s.v. $\Rightarrow f = 0$ s.v.

a $\lambda = 0 \notin \sigma_p(T)$.

- $\lambda \neq 0$: $\lambda f = Tf \Rightarrow Tf \in AC[0,1] \Rightarrow f \in AC[0,1] \Rightarrow Tf \in C^1[0,1]$

$\Rightarrow \lambda f'(t) = (Tf)' = t f(t)$, $t \in [0,1]$

$\Rightarrow f(t) = c e^{\frac{t^2}{2\lambda}}$, $t \in [0,1]$, $c \in \mathbb{C}$

$\lambda f(0) = Tf(0) = 0 \Rightarrow f(0) = 0 \Rightarrow f \equiv 0$

$\Rightarrow \lambda \notin \sigma_p(T)$

$\Rightarrow \sigma_p(T) = \emptyset$

e) T kpt, $\dim X = \infty \Rightarrow \sigma(T) = \{0\}$