Příklad 1:

a) $k<0$, $f\equiv 1 \Rightarrow \varphi_k(f) = \int_{-\pi}^{\pi} 1\,(\sin t)^k\,dt$ neexistuje, neboť $\sin t \approx t$ u 0

a t^k není u 0 integrovatelné

$k\geq 0$: $\varphi_k : X\to\mathbb{R}$ dobře definované a zjevně lineární

· $|\varphi_k(f)| \leq \int_{-\pi}^{\pi} |f(t)|\,|\sin t|^k\,dt \leq \|f\|_\infty \underbrace{\int_{-\pi}^{\pi}|\sin t|^k\,dt}_{I} = \|f\|_\infty I \Rightarrow \|\varphi_k\|\leq I$

ať $f_n : [-\pi,\pi]\to[-1,1]$ spojité splňující $f_n(t) \to \begin{cases} 1 \,..\, t\in(0,\pi] \\ 0 \,..\, t=0 \\ -1 \,..\, t\in[-\pi,0)\end{cases}$. Pak

$I \geq |\varphi_k(f_n)| = \left|\int_{-\pi}^{\pi} f_n\,(\sin t)^k\,dt\right| \xrightarrow[n\to\infty]{\text{Lebesgue}} \int_{-\pi}^{\pi}|\sin t|^k\,dt = I$

$\Rightarrow \|\varphi_k\| = I$

· $k\geq 0$ sudé: $f\equiv 1 \Rightarrow \varphi_k(f) = \int_{-\pi}^{\pi} 1\cdot(\sin t)^k\,dt = \int_{-\pi}^{\pi}|\sin t|^k\,dt = I$

$\Rightarrow$ norma se nabývá

· $k\geq 0$ liché: ať $f\in B_X$ splňuje $\varphi_k(f) = I$. Pak máme

$I = \int_{-\pi}^{\pi}\underbrace{|\sin t|^k}_{h}\,dt = \int_{-\pi}^{\pi}\underbrace{f(\sin t)^k}_{g}\,dt$, přičemž $g\leq h$.

Tedy $0 = \int_{-\pi}^{\pi}(h-g) \Rightarrow g = h$ s.v. $\Rightarrow |\sin t|^k = f(t)(\sin t)^k$ s.v.

$\Rightarrow f = \begin{cases} 1 \,..\, (0,\pi) \\ -1 \,..\, (-\pi,0)\end{cases}$ s.v. $\Rightarrow f$ není spojitá v 0 $\Rightarrow$ spor

$\Rightarrow$ norma se nenabývá

b) $k<0$, $f\equiv 1 \Rightarrow$ jako výše $\varphi_k(f)$ není definováno

$k\geq 0$: $\varphi_k : Y\to\mathbb{R}$ dobře definované a zjevně lineární

$|\varphi_k(f)| \leq \|f\|_\infty \underbrace{\int_{-\pi}^{\pi}|\sin t|^k\,dt}_{I} \Rightarrow \|\varphi_k\|\leq I$

· $f = \begin{cases} 1 \,..\, [0,\pi] \\ -1 \,..\, [-\pi,0)\end{cases} \Rightarrow \varphi_k(f) = \int_{-\pi}^{\pi} f(t)(\sin t)^k\,dt = \int_{-\pi}^{\pi}|\sin t|^k\,dt = I$

$k\geq 0$ liché

$\Rightarrow$ norma se nabývá

· $f \equiv 1 \Rightarrow \varphi_\varepsilon(f) = I \Rightarrow$ normy se nabývá

$\varepsilon \geq 0$ malé

c) $\varphi_\varepsilon : X \to \mathbb{R}$, $\varphi_\varepsilon^* : \mathbb{R} \to M[-\pi,\pi]$, kde $\mathbb{R}^* = \mathbb{R}$ pomocí

$\varepsilon \geq 0$

duality $t \in \mathbb{R} \mapsto \psi_t(x) = tx$, $x \in \mathbb{R}$, a

$X^* = M[-\pi,\pi]$ pomocí duality

$\mu \in M[-\pi,\pi] \mapsto \psi_\mu(f) = \int_{[-\pi,\pi]} f \, d\mu$, $f \in C[-\pi,\pi]$.

Pro $t \in \mathbb{R}$ a $f \in X$ platí $(\varphi_\varepsilon^*(t))(f) = \psi_t(\varphi_\varepsilon(f)) = t\,\varphi_\varepsilon(f) =$

$$= \int_{-\pi}^{\pi} f(s)\,(\sin s)^\varepsilon \cdot t \, ds = (t\psi_{\mu_0})(f) = (t\mu_0)(f).$$

$\mu_0 = (\sin s)^\varepsilon \, d\lambda(s)$

Tedy $(\varphi_\varepsilon)^*(t) = t\mu_0$, $t \in \mathbb{R}$.

Příklad 2:

b) $T_1 x = (x_1 + x_2,\ 2x_1 + 3x_2,\ 0,\ 0, \dots)$

$T_2 x = (0,\ 0,\ \tfrac{1}{4} x_4,\ \tfrac{1}{5} x_5,\ \tfrac{1}{6} x_6,\ \tfrac{1}{7} x_7, \dots)$

- $$\|T_1 x\|^2 = |x_1 + x_2|^2 + |2x_1 + 3x_2|^2 \le (|x_1| + |x_2|)^2 + 9(|x_1| + |x_2|)^2 \le$$
$$\le 10\big(|x_1|^2 + |x_2|^2 + \underbrace{2|x_1||x_2|}_{\le |x_1|^2 + |x_2|^2}\big) \le 20\,(|x_1|^2 + |x_2|^2) \le 20 \sum_{n=1}^{\infty} |x_n|^2$$

$\Rightarrow \|T_1\| \le \sqrt{20}$

- $$\|T_2 x\|^2 = \sum_{n=4}^{\infty} \left|\tfrac{1}{n} x_n\right|^2 \le \sum_{n=1}^{\infty} |x_n|^2 \Rightarrow \|T_2\| \le 1$$

$\Rightarrow T_1 + T_2 = T$ je spojitý

c) $k \ge 3 \Rightarrow T_k x := (x_1 + x_2,\ 2x_1 + 3x_2,\ \tfrac{1}{4} x_4,\ \tfrac{1}{5} x_5, \dots, \underset{\substack{\uparrow \\ k\text{-tá pozice}}}{\tfrac{1}{k+1} x_{k+1}},\ 0,\ 0, \dots)$

pak T_k je podobně jako výše spojitý a konečně dimenzionální, tedy kompaktní. Dále

$$\|T - T_k\| = \sup_{\|x\| \le 1} \|(T - T_k)x\| = \sup_{\|x\| \le 1} \left\|\left(0, \dots, 0, \tfrac{1}{k+2} x_{k+2}, \tfrac{1}{k+3} x_{k+3}, \dots\right)\right\| \le$$
$$\le \frac{1}{k+2} \sup_{\|x\| \le 1} \|x\| \le \frac{1}{k+2} \xrightarrow[k \to \infty]{} 0$$, tedy T je kompaktní.

d) $Tx = \lambda x \Rightarrow$
$$\left.\begin{aligned} x_1 + x_2 &= \lambda x_1 \\ 2x_1 + 3x_2 &= \lambda x_2 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\begin{aligned} \tfrac{1}{4} x_4 &= \lambda x_3 \\ \tfrac{1}{5} x_5 &= \lambda x_4 \\ \tfrac{1}{6} x_6 &= \lambda x_5 \\ &\ \vdots \\ \tfrac{1}{k+1} x_{k+1} &= \lambda x_k \\ &\ \vdots \end{aligned}$$

$\det\begin{pmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 1 \Rightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

- $\begin{pmatrix} 1-\lambda_1 & 1 \\ 2 & 3-\lambda_1 \end{pmatrix} = \begin{pmatrix} -1-\sqrt{3} & 1 \\ 2 & 1-\sqrt{3} \end{pmatrix}$ dává $x = \left(\frac{1}{\sqrt{3}+1}, 1, 0, 0, \dots\right)$

- $\begin{pmatrix} 1-\lambda_2 & 1 \\ 2 & 3-\lambda_2 \end{pmatrix} = \begin{pmatrix} -1+\sqrt{3} & 1 \\ 2 & 1+\sqrt{3} \end{pmatrix}$ dává $x = \left(\frac{-1}{\sqrt{3}-1}, 1, 0, 0, \dots\right)$

ať $\lambda \notin \{\lambda_1, \lambda_2\}$, pokud $\lambda = 0$, pak $x = (0, 0, 1, 0, \dots)$ je vl. vektor pro 0.

ať dále $\lambda \notin \{0, \lambda_1, \lambda_2\}$: pak $x_1 = x_2 = 0$, dále

$x_4 = 4\lambda x_3 \Rightarrow x_5 = 5 \cdot 4 \lambda^2 x_3 \Rightarrow x^6 = 6 \lambda x_5 = 6 \cdot 5 \cdot 4 \cdot \lambda^3 x_3 = \dots$

$\Rightarrow x_n = (n(n-1) \dots 4) \lambda^{n-3} x_3$

- $x_3 = 0 \Rightarrow x_4 = x_5 = x_6 = \dots = 0 \Rightarrow x = 0$

- $x_3 \neq 0 \Rightarrow x \in \ell_2 \Rightarrow \lim_{n\to\infty} |x_n| = 0 \Rightarrow \lim_{n\to\infty} ((n)(n-1)\dots 4)|\lambda|^{n-3} = 0$

tedy $\frac{n!}{6} \frac{|\lambda|^n}{|\lambda|^3} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ spor $\Rightarrow x_3 = 0$

Proto $\sigma_p(T) = \{0, \lambda_1, \lambda_2\}$

e) $\sigma(T) = \{0\} \cup \sigma_p(T) = \{0, \lambda_1, \lambda_2\}$, neboť T kptní.

a) $\{e_n : n \in \mathbb{N}\}$ je LN nekonečná množina:

$\left[\, 0 = \sum_{n=1}^{N} a_n e_n \Rightarrow 0 = \left\langle \sum_{n=1}^{N} a_n e_n, e_k \right\rangle = a_k, \quad k = 1 \dots N \,\right]$