1) $uv + xy = 1$

$v^2 + x + y^2 = 1$ v bodě $[0,0,1,1] = [x,y,u,v]$

$\Rightarrow F_1(x,y,u,v) = uv + xy - 1 \quad F_1(0,0,1,1) = 1 + 0 - 1 = 0$

$F_2(x,y,u,v) = v^2 + x + y^2 - 1 \quad F_2(0,0,1,1) = 1 + 0 + 0 - 1 = 0$, $F_1, F_2 \in C^\infty(\mathbb{R}^4)$

$\frac{dF_1}{du} = v \quad \frac{dF_1}{dv} = u$

$\frac{dF_2}{du} = 0 \quad \frac{dF_2}{dv} = 2v$ $\Rightarrow$ v bodě $[0,0,1,1]$ máme $\det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2 \neq 0$

VOIF
$\Rightarrow$ ex. $U \ni [0,0]$, $V \ni [1,1]$, že máme funkci $\varphi: U \to V$ splňující $\varphi(0,0) = [1,1]$ a $\varphi \in C^\infty(U)$ (tj. složky jsou třídy C^∞)

Píšeme $u(x,y) = \varphi_1$, $v(x,y) = \varphi_2$.

Pak na U máme:

$u_x v + u v_x + y = 0 \qquad u_y v + u v_y + x = 0$

$2v v_x + 1 = 0 \qquad 2v v_y + 2y = 0$

dosadíme $[0,0,1,1]$

$u_x \cdot 1 + 1 v_x + 0 = 0 \qquad u_y \cdot 1 + 1 \cdot v_y + 0 = 0$

$2 \cdot 1 \cdot v_x + 1 = 0 \qquad 2 \cdot 1 \cdot v_y + 0 = 0$

$v_x = -1/2 \qquad v_y = 0$

$u_x = 1/2 \qquad u_y = 0$

Tedy $Du(0,0) = (1/2, 0)$, $Dv(0,0) = (-1/2, 0)$

· $F \in C^\infty$	+1	derivace	3
· $F(0,0,1,1) = 0$	+1	dosazení	2
· $\det \neq 0$	+2	výpočet	2
· VOIF	+2	závěr	2

2) $M = \{x^2+y^2+z^2 \le 1,\ (x-1)^2+y^2+z^2 \le 1\}$

$f(x,y,z) = x+y+z$

· M je omezená v jednotkové kouli a uzavřená, M kompaktní (+1)

· $f \in C^\infty(\mathbb{R}^3)$ (+1)

· $M_1 = \{x^2+y^2+z^2<1,\ (x-1)^2+y^2+z^2<1\}$, $\nabla f = (1,1,1) \neq 0$ (+1)

· $M_2 = \{x^2+y^2+z^2=1,\ (x-1)^2+y^2+z^2<1\}$ (+3)

$\nabla g \neq 0$ na M_2

$\Rightarrow 1+\lambda x = 0$ $\quad \lambda \neq 0$

$1+\lambda y = 0 \quad \lambda(x-y)=0 \Rightarrow x=y$

$1+\lambda z = 0 \quad \lambda(z-y)=0 \Rightarrow z=y$

$\underline{x^2+y^2+z^2=1} \Rightarrow 3x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt3}$

sol $\overbrace{\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)}^{\alpha} \in M_2$

sol $\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right) \notin M_2$

· $M_3 = \{x^2+y^2+z^2<1,\ (x-1)^2+y^2+z^2=1\}$

$\nabla g \neq 0$ na M_3

$1+\lambda(x-1)=0$

$1+\lambda y = 0 \quad \lambda(y-(x-1))=0 \Rightarrow y = x-1$

$1+\lambda z = 0 \quad z=y$

$\underline{(x-1)^2+y^2+z^2=1} \Rightarrow y^2+y^2+y^2=1 \Rightarrow y = \pm\frac{1}{\sqrt3} \Rightarrow \overbrace{\left(1-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)}^{\beta} \in M_3$ (+3)

$y^2 = 1/3$ $\qquad \left(1+\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) \notin M_3$

· $M_4 = \{x^2+y^2+z^2=1,\ (x-1)^2+y^2+z^2=1\}$ (+4)

$\Rightarrow 0 = x^2-(x-1)^2 = (x-(x-1))(x+x-1) \Rightarrow x = 1/2$

$\Rightarrow$ máme $g(y,z) = 1/2+y+z$ na $N = \{y^2+z^2 = \frac34\}$, $\nabla(\text{vazby}) \neq 0$ na N

$1+\lambda y = 0$

$1+\lambda z = 0 \Rightarrow z=y$

$\underline{y^2+z^2 = \frac34} \Rightarrow 2y^2 = \frac34$

$y^2 = \frac{3}{2\cdot 4}$

$y = \pm\frac{\sqrt3}{2\sqrt2}$

$\Rightarrow \left(1/2, \frac12\sqrt{\frac32}, \frac12\sqrt{\frac32}\right) = \gamma$

$\left(1/2, -\frac12\sqrt{\frac32}, -\frac12\sqrt{\frac32}\right) = \delta$

$f(\alpha) = \sqrt3 \qquad f(\gamma) = \frac12\left(1+2\sqrt{\frac32}\right) = \frac12(1+\sqrt6) = \frac12+\sqrt{\frac32}$

$f(\beta) = 1-\sqrt3 \qquad f(\delta) = \frac12\left(1-2\sqrt{\frac32}\right) = \frac12-\sqrt{\frac32}$

(+2) $\Rightarrow \alpha$ maximum, β minimum

3) $$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1-a & 1-a^2 & 1-a^3 \\ 0 & a-1 & 1-a & a-a^2 \\ 1 & 1 & a & a^2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 0 & 2-a^2-a & 1-a^3+a-a^2 \\ 0 & a-1 & 1-a & a-a^2 \\ 1 & 1 & a & a^2 \end{array}\right)$$

$$2-a^2-a = -(a^2+a-2) = -(a+2)(a-1)$$

- $\Rightarrow$ $a = -2$ $\Rightarrow$ na prvním řádku je $0 \quad 0 \quad 0 \mid \overbrace{1-(-8)-2-4}^{3}$

  $\Rightarrow$ není řešení

- $\Rightarrow$ $a = 1$ $\Rightarrow$ $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$ má řešení $(x, y, z) = (1-s-t, s, t)$, $s, t \in \mathbb{R}$

- $\Rightarrow$ $a \notin \{1, -2\}$ $\Rightarrow$ $\left(\begin{array}{ccc|c} 0 & 0 & -(a+2)(a-1) & (1-a)(1+a)^2 \\ 0 & 1 & -1 & -a \\ 1 & 1 & a & a^2 \end{array}\right)$

$$\Rightarrow \quad z = \frac{(a+1)^2}{a+2}, \quad y = -a + \frac{(a+1)^2}{a+2} = \frac{-a^2-2a+a^2+2a+1}{a+2} = \frac{1}{a+2}$$

$$x = a^2 - az - y = a^2 - \frac{a(a+1)^2}{a+2} - \frac{1}{a+2} =$$

$$= \frac{1}{a+2}\Big(\underbrace{a^2(a+2) - a(a+1)^2 - 1}_{a^3+2a^2-a^3-2a^2-a-1}\Big) = -\frac{a+1}{a+2}$$

- úpravy: 8
- $a = -2$ 2
- $a = 1$ 2
- $a \notin \{-2, 1\}$ 3

4) $\sum (-1)^n \underbrace{\left(\sqrt[3]{n+1} - \sqrt[3]{n-1}\right)}_{a_n}$

· $a_n = \dfrac{(n+1)-(n-1)}{(n+1)^{2/3} + (n+1)^{1/3}(n-1)^{1/3} + (n-1)^{2/3}} = \dfrac{2}{n^{2/3}\left((1+\frac{1}{n})^{2/3} + (1+\frac{1}{n})^{1/3}(1-\frac{1}{n})^{1/3} + (1-\frac{1}{n})^{2/3}\right)}$

· $\sum |(-1)^n a_n| = \sum a_n$, kde $\dfrac{a_n}{\frac{1}{n^{2/3}}} \longrightarrow \dfrac{2}{1+1+1} = \dfrac{2}{3} \in (0,\infty)$.

$\sum \frac{1}{n^{2/3}}$ diverguje $\Rightarrow$ $\sum a_n$ diverguje dle srovnávacího kritéria

· $a_n = \dfrac{2}{(n+1)^{2/3} + (n^2-1)^{1/3} + (n-1)^{2/3}}$ zjevně splňuje $a_n \searrow 0$, tedy

dle Leibnize $\sum (-1)^n a_n$ konverguje

- AK: · úprava +3
  - · výběr řady 2
  - · srovnání 2
  - · závěr 2

NAK: · předpoklady Leibnize 3
  - · závěr 3