Setting $m = n = 0$ into the latter formula we obtain the *product formula for Legendre polynomials*:

$$P_l(\cos\theta_1)P_l(\cos\theta_2) = \frac{1}{\pi}\int_0^\pi P_l(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi_2)d\varphi_2.$$

2.3. Product Formulas for Functions Related to the Groups $SO_0(n,1)$ and $SO(n+1)$. Using in (31) the expressions for matrix elements of the representations T^σ of the group $SO_0(n,1)$ from Sect. 2.4, Chap. 2, we obtain the *product formula for associated Legendre functions*

$$\int_0^\pi \sinh^{-p} t\,\mathfrak{P}^{-p}_{\sigma+p}(\cosh t)C^p_m(\cos\varphi_1)\sin^{2p}\varphi_1 d\varphi_1$$
$$= \frac{(-1)^m \pi 2^{-p+1}\Gamma(\sigma+1)\Gamma(-\sigma-2p)\Gamma(m+2p)}{m!\Gamma(\sigma-m+1)\Gamma(-\sigma-m-2p)\Gamma(p)}(\sinh t_1 \sinh t_2)^{-p}$$
$$\times\, \mathfrak{P}^{-m-p}_{\sigma+p}(\cosh t_1)\mathfrak{P}^{-m-\sigma}_{\sigma+p}(\cosh t_2),$$

where $\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos\varphi_1$, $\quad p = (n-2)/2$.

Matrix elements of the representations T^l of the group $SO(n+1)$ lead to the *product formula for Gegenbauer polynomials*

$$\int_0^\pi C^p_l(\cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\psi)C_k^{p-1/2}(\cos\psi)\sin^{2p-1}\psi d\psi$$
$$= \frac{2^{2k+2p-1}\Gamma^2(p+k)(l-k)!\Gamma(2p+k-1)}{k!\Gamma(2p-1)\Gamma(l+k+2p)}$$
$$\times\,(\sin\theta\sin\varphi)^k C^{p+k}_{l-k}(\cos\theta)C^{p+k}_{l-k}(\cos\varphi).$$

2.4. Product Formulas for Bessel Functions. Applying the formula (31) to matrix elements of representations of the group $ISO(2)$ we obtain the *product formula for Bessel functions*

$$J_{n-m}(r_1)J_m(r_2) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\varphi - m\varphi_2)}J_n(r)d\varphi_2,$$

where r and φ are determined by formulas (16).

Another product formula for Bessel functions is derived with the help of matrix elements of representations of the group $ISO(n)$, $n > 2$. We have

$$J_{m+p}(r_1)J_{m+p}(r_2)(r_1r_2)^{-p}$$
$$= \frac{2^{p-1}m!\Gamma(p)}{(-1)^m\pi\Gamma(m+2p)}\int_0^\pi r^{-p}J_p(r)C^p_m(\cos\varphi)\sin^{2p}\varphi d\varphi,$$

where $r = [r_1^2 + r_2^2 + 2r_1r_2\cos\varphi]^{1/2}$, $\quad p = (n-2)/2$.

2.5. Product Formulas for Jacobi Polynomials and for Jacobi Functions. The formula (19) can be considered as the expansion of the function $P_n^{(p,q)}$ $(\cos 2\theta)$, with $\cos 2\theta$ determined by equality (17a), in Jacobi polynomials of $\cos 2\varphi$ and in Gegenbauer polynomials of $\cos\psi$. Writing down the expression for coefficients of this expansion we receive the *product formula for Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(\cos 2\theta_1)P_n^{(\alpha,\beta)}(\cos 2\theta_2) = \frac{2\Gamma(\alpha+n+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)n!}$$
$$\times \int_0^\pi\int_0^1 P_n^{(\alpha,\beta)}(\cos 2\theta)(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}dr d\psi,$$

where $\cos\varphi$ is replaced by r and $\cos 2\theta$ is determined by formula (17a).

In the same way from formula (20) we derive the *product formula for Jacobi function*

$$R_\nu^{(\alpha,\beta)}(\cosh 2t_1)R_\nu^{(\alpha,\beta)}(\cosh 2t_2) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}$$
$$\times \int_0^\pi\int_0^1 R_\nu^{(\alpha,\beta)}(2|\cosh t_1\cosh t_2 + re^{i\psi}\sinh t_1\sinh t_2|^2 - 1)$$
$$\times\,(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}dr d\psi.$$

2.6. Product Formulas for Laguerre Polynomials. We set $\sigma = -1$, $\tau = i\varphi$, $\varphi \in \mathbb{R}$, in formula (21). Then the left-hand side of this formula may be considered as the Fourier-series expansion of the function from the right-hand side. Therefore, we have the *product formula for Laguerre polynomials*

$$L_k^{m-k}(t^2)L_m^{\alpha-m}(s^2) = \frac{1}{2\pi}t^{k-m}s^{m-\alpha}\int_0^{2\pi} r^{2(\alpha-k)}$$
$$\times \exp(-tse^{i\varphi})e^{i\varphi(\alpha-m)}(t+se^{i\varphi})^{k-\alpha}L_k^{\alpha-k}(t^2+s^2+2ts\cos\varphi)d\varphi.$$

In an analogous way setting $\sigma = 1$, $\tau = i\varphi$, $\varphi\in\mathbb{R}$ in formula (22) we obtain

$$L_k^{m-k}(-t^2)L_m^{\alpha-m}(s^2) = \frac{1}{2\pi}t^{k-m}s^{m-\alpha}(-1)^{k-m}\int_0^{2\pi} r^{2(\alpha-k)}$$
$$\times \exp(tse^{i\varphi})e^{i\varphi(\alpha-m)}(t+se^{i\varphi})^{k-\alpha}L_k^{\alpha-k}(s^2-t^2-2ts\sin\varphi)d\varphi.$$

§3. Generating Functions

3.1. The General Form. Let T be a representation of the group $G = KAK$ in a Hilbert space $\mathfrak{H}$ of functions on K and let $\{f_n | n = 0,1,2,\ldots\}$ be an orthonormal basis of $\mathfrak{H}$. Then for the matrix elements $t_{mn}(h)$, $h\in A$, of this representation we have

$$t_{mn}(h) = \int_K (T(h)f_n)(k)\overline{f_m(k)}dk.$$

This equality may be considered as the formula for coefficients of expansion of the function $(T(h)f_n)(k)$ in the basis functions $f_m(k)$. Therefore,

$$(T(h)f_n)(k) = \sum_{m=0}^{\infty} t_{mn}(h)f_m(k). \tag{32}$$

This formula shows that the function $(T(h)f_n)(k)$ is a *generating function* for the matrix elements $t_{mn}(h)$, $m = 0, 1, 2, \ldots$, if it is expanded in the basis functions f_m.

For representations (18a), Chap. 1, of a semisimple noncompact Lie group formula (32) takes the form

$$\lambda(\widetilde{h}^{-1})f_n(k_h) = \sum_{m=0}^{\infty} t_{mn}(h)f_m(k), \tag{33}$$

and for representations (27), Chap. 1, of an inhomogeneous group the form

$$\exp(-\nu(\widetilde{h}))f_n(k) = \sum_{m=0}^{\infty} t_{mn}(h)f_m(k). \tag{34}$$

Writing down formulas (33) and (34) for the associated spherical functions $t_{m0}(h)$ of the representation T we have

$$\lambda(\widetilde{h}^{-1}) = \sum_{m=0}^{\infty} t_{m0}(h)f_m(k), \tag{35}$$

$$\exp(-\nu(\widetilde{h})) = \sum_{m=0}^{\infty} t_{m0}(h)f_m(k). \tag{35a}$$

3.2. Generating Functions for $\mathfrak{P}^{\tau}_{mn}(x)$. Setting $g = g_t = g(0, t, 0)$ in formula (15) of Chap. 2 and replacing $t^{\chi}_{mn}(g_t)$ by $\mathfrak{P}^{\tau}_{m'n'}(\cosh t)$ we derive the relation

$$\left(\cosh\frac{t}{2} + \sinh\frac{t}{2}e^{i\theta}\right)^{\tau+n}\left(\cosh\frac{t}{2} + \sinh\frac{t}{2}e^{-i\theta}\right)^{\tau-n} e^{-in\theta} = \sum_{m=-\infty}^{\infty} \mathfrak{P}^{\tau}_{mn}(\cosh t)e^{-im\theta}.$$

Replacing $e^{-i\theta}$ by z we have

$$\Phi(z,t) \equiv \left(z\cosh\frac{t}{2} + \sinh\frac{t}{2}\right)^{\tau+n}\left(z\sinh\frac{t}{2} + \cosh\frac{t}{2}\right)^{\tau-n} = \sum_{m=-\infty}^{\infty} \mathfrak{P}^{\tau}_{mn}(\cosh t)z^{m+\tau}. \tag{36}$$

This equality shows that $\Phi(z,t)$ is a generating function for the special functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$, $m = 0, \pm 1, \pm 2, \ldots$.

Let us take formula (36) for $\tau = \tau_1$, $m = m_1$, $n = n_1$, and then for $\tau = \tau_2$, $m = m_2$, $n = n_2$. We multiply these formulas side by side and apply expansion (36) to the left-hand side of the relation obtained. Comparing coefficients at the same powers of z we find

$$\mathfrak{P}^{\tau_1+\tau_2}_{m,n_1+n_2}(\cosh t) = \sum_{m_1=-\infty}^{\infty} \mathfrak{P}^{\tau_1}_{m_1 n_1}(\cosh t)\mathfrak{P}^{\tau_2}_{m-m_1,n_2}(\cosh t). \tag{36a}$$

In particular,

$$\mathfrak{P}^{m}_{\tau_1+\tau_2}(\cosh t) = \sum_{n=-\infty}^{\infty} \mathfrak{P}^{n}_{\tau_1}(\cosh t)\mathfrak{P}^{m-n}_{\tau_2}(\cosh t).$$

We replace $e^{i\theta}$ by z in formula (15) of Chap. 2 and then reduce the formula obtained to the form

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{1}{2\pi i}\oint_{\Gamma}\left(\cosh t + \frac{z^2+1}{2z}\sinh t\right)^{\tau-n}\left(\cosh\frac{t}{2} + z\sinh\frac{t}{2}\right)^{2n} \times z^{m-n-1}dz,$$

where Γ is the circle $|z| = a$ and $1 < a < \cosh(t/2)$. Deforming the contour Γ and replacing the variable of integration we transform this formula into

$$\mathfrak{P}^{\tau}_{mn}(\cosh t) = \frac{\sin((\tau-n)\pi)}{\pi}\int_0^{\infty} w^{\tau-n}\left(\cosh\frac{t}{2} + z\sinh\frac{t}{2}\right)^{2n} \times \frac{z^{m-n}dw}{\sqrt{w^2+2w\cosh t+1}}, \tag{37}$$

where

$$z = \frac{-w - \cosh t + \sqrt{w^2+2w\cosh t+1}}{\sinh t}. \tag{38}$$

In particular,

$$\mathfrak{P}^{\tau}_{00}(\cosh t) = \mathfrak{P}_{\tau}(\cosh t) = \frac{\sin(\tau\pi)}{\pi}\int_0^{\infty}\frac{w^{\tau}dw}{\sqrt{w^2+2w\cosh t+1}}. \tag{39}$$

Applying to equality (37) the inversion formula for Mellin transform we have

$$F(w,\cosh t) \equiv \frac{(\cosh\frac{t}{2} + z\sinh\frac{t}{2})^{2n} z^{m-n}}{\sqrt{w^2+2w\cosh t+1}} = -\frac{i}{2}\int_{a-i\infty}^{a+i\infty}\frac{\mathfrak{P}^{\tau+n}_{mn}(\cosh t)w^{-\tau-1}d\tau}{\sin(\tau\pi)},$$

where $-1 < a < m-n$ and z is determined by formula (38). This relation shows that the function $F(w, \cosh t)$ is a continual generating function for $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ with fixed m and n.

In the same way we obtain from (39) that

$$\frac{1}{\sqrt{w^2+2w\cosh t+1}} = -\frac{\mathrm{i}}{2}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{\mathfrak{P}_{\tau}(\cosh t)w^{-\tau-1}}{\sin(\tau\pi)}d\tau,$$

where $-1 < a < 0$.

If in formula (36) τ is a negative integral or half-integral number and $n < \tau$, then part of functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ vanishes. Nonvanishing functions correspond to the representation $T^{-}_{\tau=l}$ of the discrete series. Going over from the functions $\mathfrak{P}^{\tau}_{mn}(\cosh t)$ to $\mathcal{P}^{l}_{mn}(\cosh t)$ we obtain the generating function for $\mathcal{P}^{l}_{mn}(\cosh t)$:

$$\begin{aligned}\Phi(z,t) &\equiv \left(z\cosh\frac{t}{2}+\sinh\frac{t}{2}\right)^{l+n}\left(z\sinh\frac{t}{2}+\cosh\frac{t}{2}\right)^{l-n}\\ &= \sum_{m=1}^{-\infty}\left[\frac{\Gamma(l-n+1)\Gamma(-l-n)}{\Gamma(l-m+1)\Gamma(-l-m)}\right]^{1/2}\mathcal{P}^{l}_{mn}(\cosh t).\end{aligned}$$

3.3. Generating Functions for $P^l_{mn}(\cosh\theta)$. The formula (15) of Chap. 2 for integral or half-integral non-negative values of $\tau = l$ and for $|m| \le l$, $|n| \le l$ gives an integral representation of matrix elements of irreducible finite-dimensional representations of the group $SU(1,1)$. Making the appropriate analytic continuation (Sect. 2.4, Chap. 1) we obtain the integral representation of matrix elements of representations of the group $SU(2)$. Using the functions $P^l_{mn}(\cosh\theta)$ we have

$$\begin{aligned}P^l_{mn}(\cos\theta) &= \frac{\mathrm{i}^{n-m}}{2\pi}\left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}\right]^{1/2}\int_0^{2\pi}\left(\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2}\right.\\ &\left.+\mathrm{i}\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2}\right)^{l-n}\left(\mathrm{i}\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2}+\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2}\right)^{l+n}e^{\mathrm{i}m\varphi}d\varphi.\end{aligned}$$

As for the group $SU(1,1)$, we derive from here that

$$\begin{aligned}F(w,\cos\theta) &\equiv \frac{1}{\sqrt{(l-n)!(l+n)!}}\left(w\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)^{l-n}\left(\mathrm{i}w\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{l+n}\\ &= \sum_{m=-l}^{l}\mathrm{i}^{m-n}\frac{P^l_{mn}(\cos\theta)}{\sqrt{(l-m)!(l+m)!}}w^{l-m}. \qquad (40)\end{aligned}$$

Thus, $F(w,\cos\theta)$ is a generating function for the functions $P^l_{mn}(\cosh\theta)$ with fixed l and n.

The analogue of relation (37) for the functions $P^l_{mn}(x)$ is of the form

$$\begin{aligned}P^l_{mn}(\cos\theta) &= \frac{1}{2\pi\mathrm{i}}\left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}\right]^{1/2}\\ &\times\oint_{\Gamma} w^{l-n}\left(\cos\frac{\theta}{2}+\mathrm{i}t\sin\frac{\theta}{2}\right)^{2n}\frac{t^{m-n}dw}{\sqrt{w^2+2w\cosh\theta+1}},\end{aligned} \qquad (41)$$

where

$$t = \frac{w-\cos\theta+\sqrt{w^2+2w\cos\theta+1}}{\mathrm{i}\sin\theta}.$$

In order to obtain from formula (41) a generating function for $P^l_{mn}(\cosh\theta)$ we make the substitution $w = 1/h$ in the integral and use the Cauchy formula for coefficients of the Taylor series. For $|m| \le n$ we have

$$\sum_{l=n}^{\infty}\left[\frac{(l-n)!(l+n)!}{(l-m)!(l+m)!}\right]^{1/2}P^l_{mn}(\cos\theta)h^{l-n} = \frac{t^{m-n}(\mathrm{i}t\sin(\theta/2)+\cos(\theta/2))^{2n}}{\sqrt{1-2h\cos\theta+h^2}}.$$

As particular cases, we obtain from here generating functions for the associated Legendre functions $P^m_l(\cos\theta)$ and for Legendre polynomials:

$$\sum_{l=m}^{\infty}\frac{l!}{(l+m)!}P^m_l(\cos\theta)h^l = \frac{(\mathrm{i}t)^m}{\sqrt{1-2h\cos\theta+h^2}},$$

$$\sum_{l=0}^{\infty}P_l(\cos\theta)h^l = \frac{1}{\sqrt{1-2h\cos\theta+h^2}}.$$

3.4. Generating Functions for Other Special Functions. Applying formula (35) to matrix elements of the representations T^{σ} of the group $SO_0(n,1)$ from Sect. 2.4, Chap. 2, we have

$$\begin{aligned}(\cosh t-\cos\varphi\sinh t)^{\sigma} &= 2^{p+1}\Gamma(\sigma+1)\Gamma(p)\sinh^{-p}t\\ &\times\sum_{k=0}^{\infty}\frac{(-1)^k(2k-2p)}{\Gamma(\sigma-k+1)}\mathfrak{P}^{-k-p}_{\sigma+p}(\cosh t)C^p_k(\cos\varphi),\end{aligned}$$

i.e., the function $(\cosh t-\cos\varphi\sinh t)^{\sigma}$ is a generating function for the set of functions

$$\sinh^{-p}t\,\mathfrak{P}^{-k-p}_{\sigma+p}(\cosh t), \quad k = 0,1,2,\ldots,$$

under the expansion in Gegenbauer polynomials.

Applying formula (35a) to matrix elements of representations of the group $ISO(2)$ we derive that

$$e^{\mathrm{i}x\sin\theta} = \sum_{n=-\infty}^{\infty}J_n(x)e^{\mathrm{i}n\theta},$$

i.e., $e^{\mathrm{i}x\cos\theta}$ is a generating function for Bessel functions with integral index. Using formula (35a) for representations of the group $ISO(n)$, $n > 2$, we have

$$e^{itx} = \Gamma(p) \sum_{m=0}^{\infty} \mathrm{i}^m (m+p) \left(\frac{t}{2}\right)^{-p} J_{m+p}(t) C_m^p(x), \quad p = \frac{n-2}{2},$$

i.e., e^{itx} can be considered as a generating function for Bessel functions with half-integral index under the expansion in Gegenbauer polynomials.

We derive from formulas (30) and (31) of Chap. 2 that

$$c^w e^{\sigma(dz+b)} (zc+a)^\alpha = \sum_{k=0}^{\infty} c^{w+k} a^{\alpha-k} e^{\sigma b} L_k^{\alpha-k} \left(-\frac{\sigma a d}{c}\right) z^k.$$

Setting $\sigma = -1$, $c = 1$, $b = 0$, $d = x$, $a = 1$ we obtain

$$e^{-xz}(z+1)^\alpha = \sum_{k=0}^{\infty} L_k^{\alpha-k}(x) z^k, \tag{42}$$

i.e., $e^{-xz}(z+1)^\alpha$ is a generating function for $L_k^{\alpha-k}(x)$, $k = 0, 1, 2, \ldots$.

§4. Laplace Operators and Differential Equations for Special Functions

4.1. Laplace Operators. As it was mentioned in Introduction, most important differential equations of mathematical physics are invariant with respect to some transformation groups. Therefore, spaces of eigenfunctions for these operators corresponding to a fixed eigenvalue λ are carrier spaces of representations of these groups.

Differential operators commuting with transformations of a given Lie group G are constructed in the following way. We denote by $\mathfrak{U}$ the universal enveloping algebra for the Lie algebra $\mathfrak{g}$ of this group. An element Z of the algebra $\mathfrak{U}$ is called *invariant* if for all $X \in \mathfrak{g}$ we have $[X, Z] = 0$ (i.e. if Z commutes with all infinitesimal operators of the group G). One can show that all such operators are polynomials of a finite number of the operators $\Delta_1, \ldots, \Delta_k$, which will be called the *Laplace operators of the group G*.

If representations are realized by shifts in a homogeneous space $\mathcal{X}$, then differential operators of the first order correspond to operators X. Therefore, in this case the Laplace operators are differential operators of higher orders. In this set of differential operators there is an operator of the second order. It is called the *Laplace–Beltrami operator*. If $\mathcal{X}$ is a homogeneous Riemannian or pseudo-Riemannian space with a semisimple motion group G and with the invariant quadratic form $g_{\alpha\beta} dx^\alpha dx^\beta$, then the Laplace–Beltrami operator is of the form

$$\Delta = \sum_{\alpha\beta} |\det(g_{\alpha\beta})|^{-1/2} \partial_\alpha g^{\alpha\beta} |\det(g_{\alpha\beta})|^{1/2} \partial_\beta.$$

If T is an irreducible representation of the group G and Δ_k is a Laplace operator of this group, then for every $g \in G$ we have $T(g)\Delta_k = \Delta_k T(g)$. If

follows from here and from the Schur lemma that the operator Δ_k is multiple to the identity operator in the carrier space of T. Therefore, a set of numbers $(\lambda_1, \ldots, \lambda_r)$ corresponds to every irreducible representation T of the group G which are eigenvalues of the Laplace operators $\Delta_1, \ldots, \Delta_r$ of this group. For every matrix element $t_{mn}(g)$ of a representation T we have

$$\Delta_k t_{mn}(g) = \lambda_k t_{mn}(g), \quad k = 1, \ldots, r. \tag{43}$$

Representing the Laplace operators in coordinates of the group G corresponding to the Cartan decomposition $G = KAK$ we reduce equations (43) to equations for the functions $t_{mnu}(h)$ (Sect. 1.5).

4.2. The Laplace Operator on $SU(2)$. We realize the group $SU(2)$ by left-shift operators in $L^2(SU(2))$ and define the *Euler angles* on $SU(2)$: $g = g(\varphi, \theta, \psi)$. Then the infinitesimal operators A_1, A_2, A_3 corresponding to the one-parameter subgroups

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad \begin{pmatrix} \cos(\theta/2) & \mathrm{i}\sin(\theta/2) \\ \mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad \begin{pmatrix} e^{\mathrm{i}\theta/2} & 0 \\ 0 & e^{-\mathrm{i}\theta/2} \end{pmatrix},$$

are of the differential form

$$A_1 = \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\varphi} - \cot\theta \sin\psi \frac{\partial}{\partial\psi},$$

$$A_2 = -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\varphi} - \cot\theta \cos\psi \frac{\partial}{\partial\psi}, \quad A_3 = \frac{\partial}{\partial\psi}.$$

The Laplace–Beltrami operator is of the form $\Delta = A_1^2 + A_2^2 + A_3^2$. We have

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \left(\frac{\partial^2}{\partial\varphi^2} - 2\cos\theta \frac{\partial^2}{\partial\varphi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right). \tag{44}$$

4.3. The Laplace Operator on $SU(1,1)$. We realize the group $SU(1,1)$ by left-shift operators in $L^2(SU(1,1))$ and parametrize $SU(1,1)$ by the angles φ, t, ψ corresponding to the decomposition $SU(1,1) = KAK$ (formula (8), Chap. 1). If B_1, B_2, B_3 are the infinitesimal operators corresponding to the one-parameter subgroups

$$\begin{pmatrix} \cosh(\theta/2) & \sinh(\theta/2) \\ \sinh(\theta/2) & \cosh(\theta/2) \end{pmatrix}, \quad \begin{pmatrix} \cosh(\theta/2) & \mathrm{i}\sinh(\theta/2) \\ -\mathrm{i}\sinh(\theta/2) & \cosh(\theta/2) \end{pmatrix}, \quad \begin{pmatrix} e^{\mathrm{i}t/2} & 0 \\ 0 & e^{-\mathrm{i}t/2} \end{pmatrix},$$

then $\Delta = -B_1^2 - B_2^2 + B_3^2$ is the Laplace–Beltrami operator on $SU(1,1)$ and we have

$$\Delta = -\frac{1}{\sinh t} \frac{\partial}{\partial t} \sinh t \frac{\partial}{\partial t} - \frac{1}{\sinh^2 t} \left(\frac{\partial^2}{\partial\varphi^2} - 2\cosh t \frac{\partial^2}{\partial\varphi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right). \tag{45}$$

4.4. The Laplace Operator for $ISO(2)$. We realize the group $ISO(2)$ by left-shift operators in the two-dimensional real space $\mathbb{R}^2$. Then the infinitesimal operators A_1, A_2 corresponding to shifts along the coordinate axes x_1 and x_2 respectively are of differential form

$$A_1 = -\frac{\partial}{\partial x_1}, \quad A_2 = -\frac{\partial}{\partial x_2}.$$

The operator $\Delta = A_1^2 + A_2^2$ commutes with shifts from the group $ISO(2)$ and, therefore, it is the Laplace operator. In this case it coincides with the *classical Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

For the spherical system of coordinates it takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{46}$$

4.5. Differential Equations for Special Functions. Matrix elements of the irreducible representations T_l of the group $SU(2)$ in the basis $\{e^{-in\theta}\}$ are of the form

$$t^l_{mn}(g(\varphi,\theta,\psi)) = e^{-i(m\varphi+n\psi)} P^l_{mn}(\cos\theta).$$

They satisfy the differential equation (43) which in our case is

$$\Delta t^l_{mn}(g) = -l(l+1)t^l_{mn}(g).$$

Taking into account the explicit form (44) of the operator $\Delta_1 \equiv \Delta$ we obtain the differential equation for the functions $P^l_{mn}(x)$:

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{m^2+n^2-2mnx}{1-x^2}\right] P^l_{mn}(x) = -l(l+1)P^l_{mn}(x).$$

Using formula (19a) of Chap. 2 we replace the functions $P^l_{mn}(x)$ by the expressions for them in terms of Jacobi polynomials and obtain the differential equation for these polynomials:

$$\left\{(1-x^2)\frac{d}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{d}{dx} + n(n+\alpha+\beta+1)\right\} P^{(\alpha,\beta)}_n(x) = 0.$$

The matrix elements $t^\chi_{mn}(g)$ of the representations T_χ of the group $SU(1,1)$ satisfy the differential equation

$$\Delta t^\chi_{mn}(g) = \tau(\tau+1)t^\chi_{mn}(g).$$

Using formula (45) for Δ and formula (16) of Chap. 2 for $t^\chi_{mn}(g)$ we derive the differential equation for the functions $\mathfrak{P}^\tau_{mn}(x)$:

$$\left[(x^2-1)\frac{d^2}{dx^2} + 2x\frac{d}{dx} - \frac{m^2+n^2-2mnx}{x^2-1}\right]\mathfrak{P}^\tau_{mn}(x) = \tau(\tau+1)\mathfrak{P}^\tau_{mn}(x).$$

In the same way the Laplace operator (46) for the group $ISO(2)$ leads to the differential equation for Bessel functions

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right] J_n(r) = -J_n(r).$$

Using representations of the group S or S_4 we derive the differential equation for Laguerre polynomials

$$\left[x\frac{d^2}{dx^2} + (\alpha - x + 1)\frac{d}{dx} + n\right] L^\alpha_n(x) = 0.$$

Chapter 4
Representations of Lie Groups in "Continuous" Bases and Special Functions

§1. Representations of Lie Groups in "Continuous" Bases

1.1. Introductory Remarks. Up to now we considered matrix elements of group representations in orthonormal bases of carrier spaces. They allow us to study the functions $J_\nu(x)$, ${}_2F_1(\alpha,\beta;\gamma;x)$, ${}_1F_1(\alpha;\gamma;x)$ with integral or half-integral values of the parameters α,β,γ,ν. To obtain properties of these functions for arbitrary values of the parameters we have to go over to bases indexed by continuous parameters (which are analogous to the basis $\{e^{i\lambda x}\}$ of the space $L^2(\mathbb{R})$). Such bases appear when a carrier space of a representation is realized in such way that operators corresponding to an appropriate noncompact one-parameter subgroup are operators of multiplication by a function.

In this case, instead of matrix elements, we have kernels of operators acting in spaces of functions. Generally speaking, these kernels are generalized functions. We are interested in the cases when they are expressed in terms of special functions.

Unfortunately, we can not so freely use kernels as matrix elements of representations since we have to be concerned about convergence of integrals. For this reason we shall consider separate groups (as a rule, groups with simple structure) instead of classes of groups. In this case, the group-theoretical