Chapter 3
Functional Relations for Special Functions Related to Matrix Elements

§1. Addition Theorems

1.1. The General Form. Let G be a semisimple noncompact Lie group (or the corresponding compact or inhomogeneous Lie group) and let $G = KAK$ be its Cartan decomposition. If $h_1, h_2 \in A$ and $k \in K$, then there exist elements $k_1, k_2 \in K$ and $h \in A$ such that

$$h_1 k h_2 = k_1 h k_2. \tag{1}$$

Let T be a representation of the group G. Using the statements of Sect. 1.3, Chap. 2, the relation $T(h_1)T(k)T(h_2) = T(k_1)T(h)T(k_2)$ can be written as

$$\sum_{q,p} T_{ri,qp}(h_1)Q_q(k)T_{qp,sj}(h_2) = Q_r(k_1)T_{ri,sj}(h)Q_s(k_2). \tag{2}$$

It is the general form of the *addition theorem* for special functions related to the representation T.

For $k = e$ formula (2) takes the form

$$\sum_{q,k} T_{ri,qk}(h_1)T_{qk,sj}(h_2) = T_{ri,sj}(h_1 h_2). \tag{2a}$$

If the multiplicity indices i, j, p are absent in (2) (that is, if the multiplicities of representations do not exceed 1) and r, s correspond to the identity representation of the subgroup K, then formula (2) turns into the *addition theorem for the spherical functions*:

$$\sum_q t_{0q}(h_1)d_{00}^q(k)t_{q0}(h_2) = t_{00}(h), \tag{3}$$

where $d_{00}^q(k)$ is the zonal spherical function of the representation Q_q of the subgroup K.

1.2. Addition Theorems for Functions Related to the Groups $SU(1,1)$ and $SU(2)$. Elements of the subgroup A of the group $SU(1,1)$ are of the form

$$g(t) = \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix},$$

and elements of the subgroup K are of the form $k(\varphi) = \operatorname{diag}(e^{i\varphi/2}, e^{-i\varphi/2})$. We consider the decomposition $h_1 k h_2 = k_1 h k_2$, $h_1, h_2, h \in A$, $k_1, k_2, k \in K$, and denote parameters of the elements h_1 and h_2 by t_1 and t_2 respectively and of the element k by φ_2. Then the parameter t of the matrix h and the parameters φ, ψ of the matrices k_1, k_2 are determined by the formulas

$$\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos\varphi_2, \tag{4}$$

$$e^{i\varphi} = \frac{\sinh t_1 \cosh t_2 + \cosh t_1 \sinh t_2 \cos\varphi_2 + i \sinh t_2 \sin\varphi_2}{\sinh t}, \tag{5}$$

$$e^{i(\varphi+\psi)/2} = \frac{\cosh(t_1/2)\cosh(t_2/2)e^{i\varphi_2/2} + \sinh(t_1/2)\sinh(t_2/2)e^{-i\varphi_2/2}}{\cosh(t/2)}, \tag{6}$$

where $0 \le \varphi < 2\pi$, $0 \le t < \pi$, $-2\pi < \psi < 2\pi$.

The operators $T_\chi(k)$, $k = \operatorname{diag}(e^{i\varphi/2}, e^{-i\varphi/2})$, are diagonal in the basis $\{e^{-im\theta}\}$:

$$T_\chi(k)e^{-im\theta} = e^{-i(m+\varepsilon)\varphi}e^{-im\theta}.$$

The blocks $T^\chi_{ri,qk}(h)$ of the representations T_χ of $SU(1,1)$ degenerate into the usual matrix elements $t^\chi_{rq}(h)$ which are expressed in terms of $\mathfrak{P}^\tau_{rq}(\cosh t)$ (Sect. 2.1, Chap. 2). Therefore, formula (2) leads to the *addition theorem for the functions* $\mathfrak{P}^\tau_{rp}(\cosh t)$:

$$e^{-i(m\varphi+n\psi)}\mathfrak{P}^\tau_{mn}(\cosh t) = \sum_{k=-\infty}^{\infty} e^{-ik\varphi_2}\mathfrak{P}^\tau_{mk}(\cosh t_1)\mathfrak{P}^\tau_{kn}(\cosh t_2), \tag{7}$$

where the parameters are connected by formulas (4)–(6). For $m = n = 0$ it turns into the *addition theorem for Legendre functions*:

$$\mathfrak{P}_\tau(\cosh t) = \sum_{k=-\infty}^{\infty} e^{-ik\varphi_2}\mathfrak{P}^k_\iota(\cosh t_1)\mathfrak{P}^{-k}_\tau(\cosh t_2). \tag{8}$$

If τ is negative integer or half-integer, then we obtain from formula (7) the *addition theorem for the functions* $\mathcal{P}^l_{mn}(\cosh t)$ related to the discrete series representations of the group $SU(1,1)$:

$$e^{-i(m\varphi+n\psi)}\mathcal{P}^l_{mn}(\cosh t) = \sum_{k=l}^{-\infty} e^{-ik\varphi_2}\mathcal{P}^l_{mk}(\cosh t_1)\mathcal{P}^l_{kn}(\cosh t_2), \tag{9}$$

where the parameters are connected by relations (4)–(6). Considering the formula (7) for non-negative integral or half-integral $\tau \equiv l$ and for $|m| \le l$, $|n| \le l$ we receive the *addition theorem for the functions* $P^l_{mn}(\cos\theta)$ related to representations of the group $SU(2)$:

$$e^{-i(m\varphi+n\psi)}P^l_{mn}(\cos\theta) = \sum_{k=-l}^{l} e^{-ik\varphi_2}P^l_{mk}(\cos\theta_1)P^l_{kn}(\cos\theta_1), \tag{10}$$

where φ, ψ, θ, φ_2, θ_1, and θ_2 are connected by formulas (4)–(6) in which $\sinh t$, $\cosh t$, $\sinh t_j$, $\cosh t_j$, $j = 1, 2$, are replaced by $i\sin\theta$, $\cos\theta$, $i\sin\theta_j$, $\cos\theta_j$, respectively.

Due to the statements of Sects. 2.1 and 2.2, Chap. 2, formulas (9) and (10) actually are addition theorems for Jacobi polynomials.

1.3. Addition Theorems for Functions Related to the Groups $SO_0(n, 1)$ and $SO(n+1)$. Let $g(t)$ be a hyperbolic rotation in the plane $(n, n+1)$ by the angle t and let $g_1(\theta)$ be a usual rotation in the plane $(n-1, n)$ by the angle θ. The relation

$$g(t_1)g_1(\varphi)g(t_2) = g_1(\psi_1)g(t)g_1(\psi_2), \tag{11}$$

where $\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos\varphi$, is a special case of formula (1) for the group $SO_0(n,1)$. Therefore, for representations of the nonunitary principal series of the group $SO_0(n,1)$, having class 1 with respect to $SO(n)$, we have

$$\sum_k t^\sigma_{0k}(g(t_1))d^k_{00}(g_1(\varphi))t^\sigma_{k0}(g(t_2)) = t^\sigma_{00}(g(t)). \tag{12}$$

Making use of the expressions of Sect. 2.4, Chap. 2, for matrix elements and of the equality

$$t^\sigma_{0k}(g(t)) = (-1)^k t^{-\sigma-n+1}_{k0}(g(t))$$

we obtain the *addition theorem for associated Legendre functions*:

$$\begin{aligned} 2^{p-1}\Gamma(p)\Gamma(\sigma+1)\Gamma(-\sigma-2p)&\sum_{k=0}^{\infty}(-1)^k\frac{(2k+2p)}{\Gamma(\sigma-k+1)}\Gamma(-\sigma-k-2p)\\ &\times(\sinh t_1 \sinh t_2)^{-p}\mathfrak{P}^{-k-p}_{\sigma+p}(\cosh t_1)\mathfrak{P}^{-k-p}_{\sigma+p}(\cosh t_2)C^p_k(\cos\varphi)\\ &= \sinh^{-p} t\,\mathfrak{P}^{-p}_{\sigma+p}(\cosh t), \end{aligned} \tag{13}$$

where $p = (n-2)/2$ and $\cosh t = \cosh t_1 \cosh t_2 + \sinh t_1 \sinh t_2 \cos\varphi$.

Replacing the hyperbolic rotations in (11) by usual ones, in the same way with the help of the representations T^l of the group $SO(n+1)$ we obtain the *addition theorem for Gegenbauer polynomials*:

$$\begin{aligned} \frac{\Gamma(2p-1)}{\Gamma^2(p)}\sum_{m=0}^{l}&\frac{2^{2m}\Gamma^2(p+m)(l-m)!(2m+2p-l)}{\Gamma(l+m+2p)}(\sin\theta_1\sin\theta_2)^m\\ &\times C^{p+m}_{l-m}(\cos\theta_1)C^{p+m}_{l-m}(\cos\theta_2)C^{p-1/2}_m(\cos\varphi)\\ &= C^p_l(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi). \end{aligned} \tag{14}$$

where $p = (n-1)/2$.

1.4. Addition Theorems for Bessel Functions. An application of relation (1) to special functions, related to representations of the group $ISO(2)$ (Sect. 2.3, Chap. 2), leads to the *addition theorem for Bessel functions with integral index*:

$$e^{in\varphi}J_n(r) = \sum_{k=-\infty}^{\infty} e^{ik\varphi_2}J_{n-k}(r_1)J_k(r_2), \tag{15}$$

where the parameters φ and r are determined by the parameters r_1, r_2 and φ_2 according to the formulas

$$r = (r_1^2 + r_2^2 + 2r_1r_2\cos\varphi_2)^{1/2}, \quad e^{i\varphi} = \frac{r_1 + r_2e^{i\varphi_2}}{r}. \tag{16}$$

Applying formula (1) to representations of the group $ISO(n)$, $n < 2$, and using the results of Sects. 2.4 and 2.5, Chap. 2, we obtain another *addition theorem for Bessel functions*:

$$\begin{aligned} 2^p\Gamma(p)\sum_{k=0}^{\infty}(-1)^k(k+p)(r_1r_2)^{-p}J_{k+p}(r_1)J_{k+p}(r_2)C^p_k(\cos\varphi)\\ = r^{-p}J_p(r), \end{aligned} \tag{15a}$$

where $p = (n-2)/2$.

1.5. Addition Theorems for Jacobi Polynomials and Jacobi Functions. With the help of representations of the group $U(n)$ addition theorems for Jacobi polynomials are derived which differ from addition theorems (9) and (10). Every element $g \in U(n)$ is representable in the form

$$g = kh_nd_nk', \; h_n = g_{n-1}(\theta), \; k, k' \in U(n-1), \; d_n(\psi) = \mathrm{diag}(1, \ldots, 1, e^{i\psi}),$$

where $g_{n-1}(\theta)$ is the rotation in the real plane $(n-1, n)$ by the angle θ. Setting $g = g_{n-1}(\theta_1)g_{n-2}(\varphi)d_n(\psi)g_{n-1}(\theta_2)$ we have the relation

$$g_{n-1}(\theta_1)g_{n-2}(\varphi)d_n(\psi)g_{n-1}(\theta_2) = kg_{n-1}(\theta)d_n(\psi_1)k', \tag{17}$$

where k and k' are elements of $U(n-1)$ (we do not need the explicit form of them) and

$$\cos 2\theta = 2|\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\varphi e^{i\psi}|^2 - 1. \tag{17a}$$

Writing down relation (17) for operators of the irreducible representation $T^{mm'}$ of the group $U(n)$ and using the formulas of Sect. 2.8, Chap. 2, we derive the following *addition theorem for Jacobi polynomials* (Shapiro [1968], Vilenkin and Shapiro [1967]):

$$\begin{aligned} P^{(p,0)}_m(\cos 2\theta) = \sum_{k=0}^{m}\sum_{l=0}^{k} &a_{mkl}(\sin\theta_1\sin\theta_2)^{k+l}(\cos\theta_1\cos\theta_2)^{k-l}\\ &\times P^{(p+k+l,k-l)}_{m-k}(\cos 2\theta_1)P^{(p+k+l,k-l)}_{m-k}(\cos 2\theta_2)\\ &\times P^{(p-1,k-l)}_l(\cos 2\varphi)(\cos\varphi)^{k-l}\cos(k-l)\psi, \end{aligned} \tag{18}$$

where p is a non-negative integer,

$$a_{mkl} = \frac{k+p+l}{p+k}\binom{m+p+k}{m-l}\binom{m+p+l}{m-k}^{-1} \times \binom{m+p}{m}^{-1}\binom{p+k}{k}\varepsilon(k-l)$$

and $\binom{n}{k} = n!/k!(n-k)!$, $\varepsilon(k-l) = 1$ for $k = l$ and $\varepsilon(k-l) = 2$ for $k \neq l$. Differentiating both parts of relation (18) in $\cos\psi$ and taking into account the formula

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+n+1)P_n^{(\alpha+1,\beta+1)}(x)$$

we obtain

$$P_n^{(p,q)}(\cos 2\theta) = \sum_{k=0}^{n}\sum_{l=0}^{k} c_{nkl}(\sin\theta_1\sin\theta_2)^{k+l}(\cos\theta_1\cos\theta_2)^{k-l} \times (\cos\varphi)^{k-l}P_{n-k}^{(p+k+l,q+k-l)}(\cos 2\theta_1)P_{n-k}^{(p+k+l,q+k-l)}(\cos 2\theta_2) \times P_l^{(p-q-1,q+k-l)}(\cos 2\varphi)C_{k-l}^{q}(\cos\psi), \tag{19}$$

where

$$c_{nkl} = (q+k-l)(p+k+l) \times \frac{(p+q+n+k)!(p+k-1)!(q-1)!(q+n)!(n-k)!}{(p+q+n)!(p+n+l)!(q+k)!(q+n-l)!} \tag{19a}$$

(Koornwinder [1972], [1973]).

Since both sides of formula (19) are rational functions in p and q, then p, q may be replaced by $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ respectively. We simultaneously replace the factorials in (19a) by the corresponding Γ-functions.

In the same way with the help of the representations $T^{k\sigma}$ of the group $U(n-1,1)$ the *addition theorem for Jacobi functions* is derived. It is of the form

$$R_\mu^{(\alpha,\beta)}(2|\cosh t_1\cosh t_2 + r\sinh t_1\sinh t_2 e^{i\psi}|^2 - 1) = \sum_{m=0}^{\infty}\sum_{l=0}^{m} Ar^{m-l} \times (\sinh t_1\sinh t_2)^{m+l}(\cosh t_1\cosh t_2)^{m-l}R_{\mu-m}^{(\alpha+m+l,\beta+m-l)}(\cosh 2t_1) \times R_{\mu-m}^{(\alpha+m+l,\beta+m-l)}(\cosh 2t_2)P_l^{(\alpha-\beta-1,\beta+m-l)}(2r^2-1)C_{m-l}^{\beta}(\cos\psi), \tag{20}$$

where

$$A = \frac{(\alpha+m+l)(\beta+m-l)\Gamma(\alpha+\beta+\mu+m+1)\Gamma(\alpha+m)\Gamma(\beta+1)}{\beta\Gamma(\alpha+\beta+\mu+1)\Gamma(\beta+m-l+1)\Gamma(\beta+m+1)\Gamma(\mu+\alpha+1)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta+\mu+1)\Gamma(\mu+1)\Gamma(\alpha+\mu+l+1)}{\Gamma(\mu-m+1)\Gamma^2(\alpha+m+l+1)}.$$

1.6. Addition Theorems for Laguerre Polynomials. We define in the group S of triangular matrices from Sect. 1.6, Chap. 1, the one-parameter subgroups

$$g_+(t) = \begin{pmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad g_-(t) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix},\quad \varepsilon(t) = \begin{pmatrix}1 & 0 & 0\\ 0 & e^t & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$z(t) = \begin{pmatrix}1 & 0 & t\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ g_1(t) = \begin{pmatrix}1 & t & t^2/2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ g_2(t) = \begin{pmatrix}1 & -t & -t^2/2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

For $t > 0$ and $s > 0$ the relation

$$g_1(t)\varepsilon(\tau)g_1(s) = \varepsilon(\tau_1)g_1(r)\varepsilon(\tau-\tau_1)z(b)$$

is fulfilled, where $b = ts\sinh\tau$, $r^2 = t^2 + 2ts\cosh\tau + s^2$, $e^{\tau_1} = (t+se^\tau)/r$. Writing down this relation for the matrices of the representation T_χ of the group S (Sect. 2.6, Chap. 2) we have

$$\sum_{m=0}^{\infty} t^m s^{-m} e^{\tau m} L_k^{m-k}(-\sigma t^2)L_m^{\alpha-m}(-\sigma s^2) = t^k s^{-\alpha}\exp(\sigma t s e^\tau + \tau\alpha)r^{2(\alpha-k)}(t+se^\tau)^{k-\alpha}L_k^{\alpha-k}(-\sigma r^2), \tag{21}$$

where $|t/s| < 1$.

From equality

$$g_1(t)\varepsilon(\tau)g_2(s) = \varepsilon(\tau_1)g_2(r)\varepsilon(\tau-\tau_1)z(b),$$

where $b = ts\cosh\tau$, $r^2 = s^2 - t^2 - 2ts\sinh\tau$, $e^{\tau_1} = (t+se^\tau)/r$, we obtain the *addition theorem*

$$\sum_{m=0}^{\infty} t^m(-s)^{-m}e^{\tau m}L_k^{m-k}(-\sigma t^2)L_m^{\alpha-m}(\sigma s^2) = (-1)^k t^k s^{-\alpha}\exp(\sigma t s e^\tau + \tau\alpha)r^{2(\alpha-k)}(t+se^\tau)^{k-\alpha}L_k^{\alpha-k}(\sigma r^2). \tag{22}$$

The equality

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_-(r)z(b),$$

where $r = (t^2-s^2)/s$, $b = (t^2-s^2)/2$, leads to the formula

$$\sum_{m=0}^{\infty}(-1)^m L_k^{m-k}(-\sigma t^2)L_m^{n-m}(-\sigma s^2) = \frac{\sigma^k(-\sigma)^{-n}}{(k-n)!}e^{-\sigma s^2}(t^2-s^2)^{k-n},$$

where $k \geq n$. If $k < n$, then the sum in this formula is equal to zero.

For $s > t > 0$ and $e^\tau = t/s$ we have the equality

$$g_1(t)\varepsilon(\tau)g_1(-s) = \varepsilon(\tau)g_+(-r)z(b),$$

where $r = (s^2 - t^2)/s$, $b = (s^2 - t^2)/2$. We obtain from here that

$$\sum_{m=0}^{\infty}(-1)^m t^{2m} s^{-2m} L_k^{m-k}(-\sigma t^2) L_m^{n-m}(-\sigma s^2) = \frac{(-1)^k n!}{k!(n-k)!} t^{2k} s^{-2n}(s^2-t^2)^{n-k} e^{-\sigma t^2},$$

where $n \geq k$. If $n < k$, then this sum is equal to zero.

1.7. The Addition Theorem for Hermite Polynomials. In the group S_3 the equality

$$g(0, r_1, s_1) g(0, r_2, s_2) = g(0, r_1 + s_1, r_2 + s_2)$$

is fulfilled. We set $s_1 = z\sqrt{2r_1}$, $s_2 = w\sqrt{2r_2}$, $r_1 = \cos^2 t$, $r_2 = \sin^2 t$, and write it down for matrices of the representations T_α. Taking into account formulas of Sect. 2.7, Chap. 2, we receive the *addition theorem for Hermite polynomials*:

$$H_n(z \sin t + w \cos t) = \sum_{k=0}^{n} \binom{n}{k} \sin^k t \cos^{n-k} t H_k(z) H_{n-k}(w), \tag{23}$$

where $\binom{n}{k} = n!/k!(n-k)!$.

1.8. Recurrence Relations. Some of the recurrence relations for special functions are infinitesimal forms of addition theorems.

Example 1. The formula (10) for $\varphi_2 = 0$ takes the form

$$\sum_{k=-l}^{l} P^l_{mk}(\cos\theta_1) P^l_{kn}(\cos\theta_2) = P^l_{mn}(\cos(\theta_1 + \theta_2)).$$

We differentiate this equality in θ_2 and put $\theta_2 = 0$. Since $\frac{d}{d\theta} P^l_{mn}(\cos\theta)\big|_{\theta=0} = 0$ for $m \neq n \pm 1$ and

$$\frac{d}{d\theta} P^l_{n+1,n}(\cos\theta)\Big|_{\theta=0} = \frac{1}{2}\sqrt{(l-n)(l+n+1)},$$

$$\frac{d}{d\theta} P^l_{n-1,n}(\cos\theta)\Big|_{\theta=0} = -\frac{1}{2}\sqrt{(l+n)(l-n+1)},$$

then replacing $\cos\theta_1$ by x we obtain the recurrence relation

$$\sqrt{1-x^2}\frac{d}{dx} P^l_{mn}(x) = \frac{1}{2}\Big[\sqrt{(l+n)(l-n+1)} P^l_{m,n-1}(x) - \sqrt{(l-n)(l+n+1)} P^l_{m,n+1}(x)\Big]. \tag{24}$$

Example 2. Setting $\varphi_2 = \pi/2$ in formula (10) we have

$$e^{-i(m\varphi + n\psi)} P^l_{mn}(\cos\theta) = \sum_{k=-l}^{l} i^{-k} P^l_{mk}(\cos\theta_1) P^l_{kn}(\cos\theta_2), \tag{25}$$

where

$$\cos\theta = \cos\theta_1 \cos\theta_2, \qquad e^{i\varphi} = \frac{\sin\theta_1 \sin\theta_2 + i\sin\theta_2}{\sin\theta},$$

$$e^{i(\varphi+\psi)/2} = \frac{\sqrt{2}\cos((\theta_1+\theta_2)/2) + i\cos((\theta_1-\theta_2)/2)}{2\cos(\theta/2)}.$$

Differentiating both sides of relation (25) in θ_2 and setting $\theta_2 = 0$, applying transformations and replacing $\cos\theta_1$ by x we obtain

$$\frac{m - nx}{\sqrt{1-x^2}} P^l_{mn}(x) = \frac{1}{2}[\sqrt{(l+n)(l-n+1)} P^l_{m,n-1}(x) + \sqrt{(l-n)(l+n+1)} P^l_{m,n+1}(x)]. \tag{26}$$

Other recurrence relations can be derived with the help of Clebsch–Gordan coefficients of group representations (Vilenkin [1965b], Sect. 8, Chap. 3).

1.9. Recurrence Relations and Differential Equations for Special Functions. To derive the second order differential equations which are satisfied by special functions, one chooses recurrence relations such that their successive action on special function leads to a multiplication of it by a number. Recurrence relations raising and lowering one of the indices of a special function are used for this derivation.

Example 3. The recurrence formulas (24) and (26) are equivalent to the relations

$$\left[\sqrt{1-x^2}\frac{d}{dx} + \frac{nx-m}{\sqrt{1-x^2}}\right] P^l_{mn}(x) = -\sqrt{(l-n)(l+n+1)} P^l_{m,n+1}(x),$$

$$\left[\sqrt{1-x^2}\frac{d}{dx} - \frac{nx-m}{\sqrt{1-x^2}}\right] P^l_{mn}(x) = \sqrt{(l+n)(l-n+1)} P^l_{m,n-1}(x).$$

They lead to the relation

$$\left[\sqrt{1-x^2}\frac{d}{dx} - \frac{(n+1)x-m}{\sqrt{1-x^2}}\right]\left[\sqrt{1-x^2}\frac{d}{dx} + \frac{nx-m}{\sqrt{1-x^2}}\right] P^l_{mn}(x) = -(l-n)(l+n+1) P^l_{mn}(x).$$

Removing the parantheses, after simplification we obtain the differential equation for the functions $P^l_{mn}(x)$:

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{m^2+n^2-2mnx}{1-x^2}\right] P^l_{mn}(x) = -l(l+1) P^l_{mn}(x).$$

Differential equations for special functions are also derived with the help of Laplace operators (Sect. 4.5 below).

1.10. Orthogonality Relations. Matrix elements of irreducible representations of a compact group satisfy the orthogonality relation

$$\int_G t^{\chi}_{mn}(g)\overline{t^{\psi}_{kl}(g)}dg = (\dim T_{\chi})^{-1}\delta_{\chi\psi}\delta_{mk}\delta_{nl}. \tag{27}$$

We assume that matrix elements are taken with respect to an orthogonal basis $\{e_n\}$ which agrees with a decomposition of restrictions of representations of the group G onto the subgroup K. We represent g as $g = khk'$, $k, k' \in K$, $h \in A_k$, decompose the matrix elements $t^{\chi}_{mn}(g)$ into a sum of products of matrix elements for k, h, k', and integrate with respect to k and k'. Due to the orthogonality of matrix elements of representations of the subgroup K and due to decomposition (3), Chap. 1, of the measure dg, we derive the orthogonality relation for the matrix elements $t^{\chi}_{mn}(h)$ from Sect. 1.5, Chap. 2:

$$\sum_j (\dim S_j)\int_{A_k} t^{\chi}_{mnj}(h)\overline{t^{\psi}_{mnj}(h)}\mu(h)dh = \frac{(\dim Q_m)(\dim Q_n)}{\dim T_{\chi}}\delta_{\chi\psi}. \tag{28}$$

In particular, for functions $t^{\chi}_{m0}(g)$ we have

$$\int_{A_k} t^{\chi}_{m0}(h)\overline{t^{\psi}_{m0}(h)}\mu(h)dh = \frac{\dim Q_m}{\dim T_{\chi}}\delta_{\chi\psi}. \tag{29}$$

For the group $SU(2)$ relation (28) takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} t^{l}_{mn}(\theta)t^{l'}_{mn}(\theta)\sin\theta d\theta = (\dim T_l)^{-1}\delta_{ll'}.$$

Taking into account the connection of the matrix elements $t^{l}_{mn}(\theta)$ with Jacobi polynomials we find that, for fixed α and β, the system of polynomials

$$2^{-(\alpha+\beta+1)/2}\left[\frac{n!(n+\alpha+\beta)!(\alpha+\beta+2n+1)}{(n+\alpha)!(n+\beta)!}\right]^{1/2} P^{(\alpha,\beta)}_n(x), \quad n = 0, 1, 2, \ldots,$$

is orthonormal on the interval [-1,1] with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$.

Writing down relation (29) for matrix elements of the representations T^l of the group $SO(n)$ we derive the orthogonality relation for Gegenbauer polynomials. The orthogonality relation for Laguerre polynomials is connected with representations of the group S_4.

§2. Product Formulas

2.1. The General Formulation. We use in formula (2) the subblocks $T^{ri,sj}_{uk,vm}(g)$ instead of the blocks $T_{ri,sj}(g)$, write it for matrix elements, multiply its both sides by the matrix element $\overline{d^{q}_{u\beta,v\delta}(k)}$, and integrate over the subgroup K. Due to the orthogonality relation for matrix elements, we obtain the relation

$$t_{rqu}(h_1)t_{qsv}(h_2) = \sqrt{\dim Q_q}\sum_{w,\gamma}\int_K t_{rsw}(h)d^{r}_{u\beta,w\gamma}(k_1) \times \overline{d^{q}_{u\beta,v\delta}(k)}d^{s}_{w\gamma,v\delta}(k_2)dk. \tag{30}$$

(Recall that h, k_1 and k_2 are functions of the element k of the subgroup K.)

If Q_r and Q_s are the identity (unit) representations of K, then formula (30) turns into the *product formula for associated spherical functions*

$$t_{0q}(h_1)t_{q0}(h_2) = \sqrt{\dim Q_q}\int_K t_{00}(h)\overline{d^{q}_{00}(k)}dk. \tag{31}$$

2.2. Product Formulas for Functions Related to the Groups $SU(1,1)$ and $SU(2)$. Using in (30) the expressions for matrix elements of the representations T_{χ} of the group $SU(1,1)$ from Sect. 2.1, Chap. 2, we obtain the *product formula for the functions* $\mathfrak{P}^{\tau}_{mn}(\cosh t)$:

$$\mathfrak{P}^{\tau}_{mk}(\cosh t_1)\mathfrak{P}^{\tau}_{kn}(\cosh t_2) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k\varphi_2 - m\varphi - n\psi)}\mathfrak{P}^{\tau}_{mn}(\cosh t)d\varphi_2.$$

It leads to the *product formula for Legendre functions and for associated Legendre functions*

$$\mathfrak{P}_{\tau}(\cosh t_1)\mathfrak{P}_{\tau}(\cosh t_2) = \frac{1}{2\pi}\int_0^{2\pi}\mathfrak{P}_{\tau}(\cosh t_1\cosh t_2 + \sinh t_1\sinh t_2\cos\varphi_2)d\varphi_2,$$

$$\mathfrak{P}^{k}_{\tau}(\cosh t_1)\mathfrak{P}^{-k}_{\tau}(\cosh t_2) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\varphi_2}\mathfrak{P}_{\tau}(\cosh t_1\cosh t_2 + \sinh t_1\sinh t_2\cos\varphi_2)d\varphi_2.$$

Representations of the discrete series of the group $SU(1,1)$ and the representations T_l of the group $SU(2)$ lead to the *product formulas for Jacobi polynomials*

$$\mathcal{P}^{l}_{mk}(\cosh t_1)\mathcal{P}^{l}_{kn}(\cosh t_2) = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\varphi_2 - m\varphi - n\psi)\mathcal{P}^{l}_{mn}(\cosh t)d\varphi_2,$$

$$P^{l}_{mk}(\cos\theta_1)P^{l}_{kn}(\cos\theta_2) = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\varphi_2 - m\varphi - n\psi)P^{l}_{mn}(\cos\theta)d\varphi_2.$$