one continuous parameter τ) on the spaces (3a-c) in the coordinate system corresponding to the subgroup K we have separation of variables:

$$t^{\chi}_{M0}(kg(\alpha)) = t^{m}_{M'0}(k)t^{\chi}_{m0}(g(\alpha)).$$

It turns out that the function $t^{\chi}_{m0}(g(\alpha))$ for the symmetric space (3c) coincides with the corresponding function $t^{\chi'}_{r0}(g(\alpha))$ of the space $SU(2p,2q)/S(U(2p,2q-1)\times U(1))$, and the function $t^{\chi}_{m0}(g(\alpha))$ for the space (3b) coincides with the corresponding function $t^{\chi'}_{r0}(g(\alpha))$ of the space $SO_0(2p,2q)/SO_0(2p,2q-1)$ (Vilenkin and Klimyk [1985b]). This reduces harmonic analysis on the spaces (3b) and (3c) to that on the space (3a) with even p and q. In particular, there is a correspondence between representations of the discrete square integrable series on the spaces

$$Sp(p,q)/Sp(p,q-1)\times Sp(1) \quad \text{and} \quad SO_0(4p,4q)/SO_0(4p,4q-1),$$

as well as between representations of the discrete series on the spaces

$$SU(p,q)/S(U(p,q-1)\times U(1)) \quad \text{and} \quad SO_0(2p,2q)/SO_0(2p,2q-1)$$

(Vilenkin and Klimyk [1987]).

The functions $t^{\chi}_{m0}(g(\alpha))$ for the space $X = SO_0(p,q)/SO_0(p,q-1)$ are expressed in terms of the matrix elements $\mathfrak{P}^{\sigma}_{mn}(\cosh t)$ of representations of the group $SU(1,1) \sim SO_0(2,1)$:

$$t^{\chi}_{(kk')0}(g(\alpha)) = c(\tanh\alpha)^{1-p/2}(\cosh\alpha)^{-(p+q-4)/2}\mathfrak{P}^{\sigma}_{\tau\tau'}(\cosh 2\alpha),$$

where k and k' correspond to the highest weights $(k,0,\ldots,0)$ and $(k',0,\ldots,0)$ of the representations of the groups $SO(p)$ and $SO(q)$, c is independent on α and is related to the Plancherel measure on $L^2(X)$, and

$$\sigma = \tau + \frac{p+q-4}{2}, \quad \tau = \frac{k+k'}{2} + \frac{p+q-4}{2}, \quad \tau' = \frac{k-k'}{2} - \frac{p-q}{2}$$

(Vilenkin and Klimyk [1987]). Here τ is the number characterizing χ. Thus, harmonic analysis on $SO_0(p,q)/SO_0(p,q-1)$ is related to that on the group $SU(1,1)$. This fact seems to admit a generalization onto pseudo-Riemannian spaces of higher rank, i.e., harmonic analysis on a space G/H of rank r is connected with that on some simple group G' of real rank r.

Chapter 1
Representations of Lie Groups Relating to Special Functions

§1. Decompositions of Groups

1.1. Iwasawa and Cartan Decompositions. We assume that the reader is familiar with the principal concepts of the theory of Lie groups, Lie algebras and their representations. As a rule, we consider the classical complex Lie groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ and their compact and noncompact real forms, as well as the groups which are "triple" to some pairs of Cartan dual real groups; in particular, groups of inhomogeneous linear transformations.

For the construction of representations of such groups and for studying properties of their matrix elements we shall need certain factorizations of groups into products of subgroups. We give here these factorizations in a convenient form. Let G be a connected noncompact real linear semisimple Lie group and let K be its maximal compact subgroup. We denote by $\mathfrak{g}$ the Lie algebra of the group G and by $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to K. An involutive automorphism θ exists in $\mathfrak{g}$ for which $\mathfrak{k}$ is the stationary subspace. The subspace $\{X \mid \theta X = -X\}$ of $\mathfrak{g}$ is denoted by $\mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is transformed by the exponential map $\mathfrak{g} \to G$ into the decomposition $G = K\mathcal{P}$ of the group G where $\mathcal{P} = \exp\mathfrak{p}$.

The Killing–Cartan form

$$B(X,Y) = \mathrm{Tr}\,(\mathrm{ad}\,X)(\mathrm{ad}\,Y), \quad X, Y \in \mathfrak{g},$$

where $(\mathrm{ad}\,X)Z = [X,Z]$, defines a symmetric bilinear form on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is semisimple if and only if this form is nondegenerate. We have $B(X,X) < 0$ on $\mathfrak{k}$ and $B(X,X) > 0$ on $\mathfrak{p}$. Consequently,

$$\langle X, Y\rangle = -B(X,\theta Y) \tag{1}$$

is a strictly positive definite scalar product on $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal commutative subalgebra in $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is said to be the *real rank* of $\mathfrak{g}$ and of G. The subgroup $A = \exp\mathfrak{a}$ is commutative.

The operators $\mathrm{ad}\,H$, $H \in \mathfrak{a}$, are skew-Hermitian with respect to the scalar product (1) and, therefore,

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma} \mathfrak{g}_{\gamma}, \tag{2}$$

where $\mathfrak{g}_0$ is the kernel of the operator $\mathrm{ad}\,H$ and $\mathfrak{g}_{\gamma}$ correspond to the eigenvalues $\gamma(H)$, $H \in \mathfrak{a}$. The decomposition (2) is orthogonal. The linear forms γ are called the *restricted roots* of the pair $(\mathfrak{g},\mathfrak{a})$, and the subspaces $\mathfrak{g}_{\gamma}$ are called the *root spaces*.

If $H_1, \ldots, H_l$ is a basis of $\mathfrak{a}$ and the first non-zero number in the sequence $\{\gamma(H_1), \ldots, \gamma(H_l)\}$ is positive (negative), then the root γ is said to be *positive (negative)* with respect to this basis. The dimension of $\mathfrak{g}_\gamma$ is called the *multiplicity* of the root γ and is denoted by $m(\gamma)$. The half-sum of the positive restricted roots with multiplicities is denoted by ρ:

$$\rho = \frac{1}{2}\sum_{\gamma>0} m(\gamma)\gamma. \tag{2a}$$

The sum $\mathfrak{n} = \sum_{\gamma>0}\mathfrak{g}_\gamma$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and $N = \exp\mathfrak{n}$ is a maximal nilpotent subgroup of G. The group G has an Iwasawa decomposition $G = KAN$, which means that any element $g \in G$ is uniquely representable in the form $g = khn$ where $k \in K$, $h \in A$, $n \in N$. Moreover, the mapping $(k, h, n) \to khn$ is an analytic diffeomorphism of $K \times A \times N$ onto G.

Let M be the centralizer of the subgroup A in K. The subgroup $P = MAN$ is called a *minimal parabolic subgroup* of G. A subgroup P', which contains P and is different from G, is called a *parabolic subgroup*. Parabolic subgroups P' are obtained from $P = MAN$ by extension of the compact subgroup M, that is, $P' = M'AN$ where $M \subset M' \subset K$. Every parabolic subgroup P' has maximal semisimple subgroup which is uniquely determined. Using this semisimple subgroup we can represent P' in the form $P' = H'A'N'$, where $A' \subset A$, $N' \subset N$ and H' is the reductive subgroup for which $H' \cap A' = H' \cap N' = \{e\}$.

The factorization $G = KAK$ of the group G is called the *Cartan decomposition* of G. We have

$$khk_1 = k'h'k_1', \quad k, k_1, k', k_1' \in K, \quad h, h' \in A,$$

if $h = h'$, $k = k'm$, $k_1 = m^{-1}k_1'$, $m \in M$. To obtain a unique decomposition, one has to take the subset $A^+ = \exp\mathfrak{a}^+$ instead of A where $\mathfrak{a}^+$ is the set of elements H from $\mathfrak{a}$ such that $\gamma(H) > 0$ for all restricted roots γ of the pair $(\mathfrak{g}, \mathfrak{a})$. The set KA^+K is everywhere dense in G.

Let $\bar{\mathfrak{n}} = \sum_{\gamma<0}\mathfrak{g}_\gamma$ and $\bar{N} = \exp\bar{\mathfrak{n}}$. Then for almost all $g \in G$ we have $g = n_1 mhn$ where $n_1 \in \bar{N}$, $m \in M$, $h \in A$, $n \in N$. Therefore, the equality $G = \bar{N}MAN$ is valid almost everywhere. It is called the *Gauss decomposition*.

Let G_c be the complexification of G, and let G_k be the compact real form of the group G_c. If A_c is the complexification of the subgroup A, then $A_k = A_c \cap G_k$ is a commutative subgroup of G_k. We have the decomposition $G_k = KA_kK$. It is dual to the decomposition $G = KAK$. If $A = \exp\mathfrak{a}$, then $A_k = \exp i\mathfrak{a}$, $i = \sqrt{-1}$.

Factorizations of invariant measures dg on G and G_k are associated with the Cartan decompositions of these groups. If $g = khk'$ where $h \in A$ or $h \in A_k$, then

$$dg = \mu(h)\, dk\, dh\, dk', \tag{3}$$

where dk and dh are the invariant measures on K and on A or A_k, respectively. The multiplier $\mu(h)$ is defined by the formula

$$\mu(h) = \prod_{\gamma>0}\left[\sinh\frac{(\gamma, H)}{2}\right]^{m(\gamma)}, \quad h \in \exp H, \quad H \in \mathfrak{a}, \tag{4}$$

for the noncompact group G and by the formula

$$\mu(h) = \prod_{\gamma>0}\left[\sin\frac{(\gamma, H)}{2}\right]^{m(\gamma)}, \quad h = \exp iH, \quad H \in \mathfrak{a}, \tag{5}$$

for the compact group G_k. The products in (4) and (5) are over all positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and (γ, H) is the value of γ at H.

1.2. Decompositions of the Group $SL(2,\mathbb{R})$. The subgroups $K, A, N, \bar{N}$ of this group consist of the matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

respectively. We also have $M = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$. The subgroups K, A and N (or $\bar{N}$) are said to be *elliptic, hyperbolic and parabolic*, respectively.

For $SL(2,\mathbb{R})$ we have the following decompositions

$$SL(2,\mathbb{R}) = KAN = KNA = NKA = KAK = \bar{N}AN. \tag{6}$$

The order of the subgroups may also be reversed. The decomposition

$$SL(2,\mathbb{R}) = NAN \cup NsAN, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

is also used in the theory of special functions.

The group $SU(1,1)$ is often used instead of $SL(2,\mathbb{R})$. These groups are isomorphic. Elements $g \in SU(1,1)$ are representable in the form

$$g \equiv g(\varphi, t, \psi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix}\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \tag{8}$$

(the Cartan decomposition). The Cartan dual group to $SU(1,1)$ is $SU(2)$. The Cartan decomposition for its elements is

$$g \equiv u(\varphi, \theta, \psi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}. \tag{9}$$

1.3. Decompositions of the Groups $SO_0(n,1)$ and $SO(n+1)$. For these groups $K = SO(n)$ and the subgroups A and A_k consist of the matrices

$$g_n'(t) = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \quad g_n(\theta) = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -rm\sin\theta & \cos\theta \end{pmatrix}, \tag{10}$$

where I_{n-1} is the identity $(n-1)\times(n-1)$ matrix. The subgroup M is isomorphic to $SO(n-1)$. The subgroup N consists of the matrices

$$n(\mathbf{a}) = \begin{pmatrix} E_{n-1} & \mathbf{a}^t & -\mathbf{a}^t \\ \mathbf{a} & 1 + \frac{(\mathbf{a},\mathbf{a})}{2} & -\frac{(\mathbf{a},\mathbf{a})}{2} \\ \mathbf{a} & \frac{(\mathbf{a},\mathbf{a})}{2} & 1 - \frac{(\mathbf{a},\mathbf{a})}{2} \end{pmatrix},$$

where $\mathbf{a} = (a_1, \ldots, a_{n-1})$, $a_j \in \mathbb{R}$, and $(\mathbf{a},\mathbf{a}) = \sum_{j=1}^{n-1} a_j^2$. One can directly verify that N is a commutative group.

Let $E_{n,1}$ be the real pseudo-Euclidean space with the bilinear form

$$[\mathbf{x}, \mathbf{y}] = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}.$$

The Riemannian symmetric space $SO_0(n,1)/SO(n)$ is identified with the upper sheet of the hyperboloid $H_n = \{\mathbf{x} \in E_{n,1} \mid [\mathbf{x},\mathbf{x}] = 1\}$, and $K \equiv SO(n)$ is the isotropy subgroup at the point $\mathbf{x}_0 = (0, \ldots, 0, 1) \in H_n$.

Let $\xi_0 = (0, \ldots, 0, 1, 1) \in E_{n,1}$. Then we have

$$M = \{k \in SO(n) \mid k\,\xi_0 = \xi_0\}, \quad MN = \{g \in SO_0(n,1) \mid g\,\xi_0 = \xi_0\},$$

and the space $SO_0(n,1)/MN$ is identified with the upper sheet of the cone $\{\mathbf{x} \in E_{n,1} \mid [\mathbf{x},\mathbf{x}] = 0,\ \mathbf{x} \neq \mathbf{0}\}$.

Since $[g\mathbf{x}, g\mathbf{y}] = [\mathbf{x},\mathbf{y}]$ for $g \in SO_0(n,1)$, and

$$g_n'(t)\,\xi_0 = (0, \ldots, 0, e^t, e^t),$$

then for $k \in K$ and $n \in N$ we have

$$[\mathbf{x}_0, k\,g_n'(t) n\,\xi_0] = [\mathbf{x}_0, g_n'(t)\,\xi_0] = e^t.$$

Thus, the parameter t of the element $g_n'(t) \in A$ from the Iwasawa decomposition $g = k\,g_n(t)\,n$, $g \in SO_0(n,1)$, is defined by the formula

$$t = \log\,[\mathbf{x}_0, g\,\xi_0]. \tag{10a}$$

1.4. Decompositions of the Groups $U(n,1)$ and $U(n+1)$. For these groups we have $K = U(n-1) \times U(1)$. Instead of K it is convenient to use the subgroup $K' = U(n-1)$. The one-parameter subgroup of the diagonal matrices $\mathrm{diag}\,(1, \ldots, 1, e^{i\varphi}, 1, \ldots, 1)$, where $e^{i\varphi}$ is situated on the j-th place, is denoted by D_j. If A and A_k are the subgroups of matrices (10), then the decompositions

$$U(n,1) = K' D_{n+1} A K', \quad U(n+1) = K' D_{n+1} A_k K' \tag{11}$$

hold. The subgroup of matrices $g \in U(n+1)$ with determinant one is denoted by $SU(n+1)$.

1.5. Inhomogeneous Lie Groups. A third group G_s is associated with the dual compact and noncompact semisimple Lie groups G_k and G, constructed in the following way. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G, as in Sect. 1.1. Since $\mathfrak{p}$ is the eigenspace of the involutive automorphism θ corresponding to the eigenvalue -1, then $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Therefore, the correspondence $X \to \mathrm{ad}\,X$ defines an action of the subalgebra $\mathfrak{k}$ in the space $\mathfrak{p}$. The corresponding action of the subgroup K in $\mathfrak{p}$ is denoted by Ad. The space $\mathfrak{p}$ is equipped with the scalar product (1) which is invariant with respect to Ad. The group G_s is the semidirect product $G_s = K \times \mathfrak{p}$ of the compact group K with the vector invariant subgroup $\mathfrak{p}$. Its elements are multiplied as

$$(k,p)(k',p') = (kk', (\mathrm{Ad}\,k)p' + p).$$

The elements $(k,p) \in G_s$ are usually represented in the matrix form:

$$(k,p) \longrightarrow \begin{pmatrix} \mathrm{Ad}\,k & p \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{a}$ and $\mathfrak{a}^+$ be as in Sect. 1.1. From the action of the operators $\mathrm{Ad}\,k$, $k \in K$, on $H \in \mathfrak{a}^+$ we obtain the orbit $\mathcal{O}_H$ in $\mathfrak{p}$. The orbits $\mathcal{O}_H$ and $\mathcal{O}_{H'}$ are nonoverlapping if $H \neq H'$. The set $\{(\mathrm{Ad}k)\ \mathfrak{a}^+ \mid k \in K\}$ is everywhere dense in $\mathfrak{p}$.

From the equations $(\mathrm{Ad}k)\mathfrak{a}^+ = k\mathfrak{a}^+k^{-1}$ and $G_s = K \times \mathfrak{p}$ we obtain for G_s the analogue of the Cartan decomposition

$$G_s = (K,0)(e,\overline{\mathfrak{a}^+})(K,0), \tag{12}$$

where $\overline{\mathfrak{a}^+}$ is the closure of the set $\mathfrak{a}^+$.

Let $\mathfrak{t}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. For G_s the analogue of the Iwasawa decomposition is

$$G_s = (K,0)(e,\mathfrak{a})(e,\mathfrak{t}). \tag{12a}$$

The subgroup

$$P_s = (M,0)(e,\mathfrak{a})(e,\mathfrak{t}) \tag{13}$$

of G_s corresponds to the minimal parabolic subgroup $P = MAN$ of G.

If $G = SO_0(n,1)$, then G_s coincides with the inhomogeneous rotation group $ISO(n)$. It consists of the matrices

$$\begin{pmatrix} k & \mathbf{a} \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad k \in SO(n), \quad \mathbf{a} = (a_1, \ldots, a_n)^t.$$

The group $ISO_0(n,1)$ is defined analogously. It consists of the matrices

$$\begin{pmatrix} h & \mathbf{a} \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad h \in SO_0(n,1), \quad \mathbf{a} = (a_1, \ldots, a_{n+1})^t.$$

This group is called the $(n+1)$-*dimensional Poincaré group*.

1.6. The Groups S and S_j. The group G_s is obtained from the groups $G = K\mathcal{P}$ and $G_k = K\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{P}} = \exp \mathrm{i}\mathfrak{a}$, $\mathrm{i} = \sqrt{-1}$, by "geometric rectification" of the spaces $\mathcal{P}$ and $\widetilde{\mathcal{P}}$. Repeating the "rectification" operation in the subgroup K and in the subsequent subgroups, we obtain groups of triangular or block–triangular matrices. The group S of the matrices

$$g \equiv g(a,b,c,d) = \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b,c,d \in \mathbb{C}, \quad c \neq 0, \tag{14}$$

is the simplest "rectified" group. It contains the subgroup S_4 consisting of the matrices

$$s \equiv s(w,\alpha,\delta) = g\left(e^{-\mathrm{i}\alpha}\frac{\overline{w}}{2},\ \mathrm{i}\delta - \frac{|w|^2}{2},\ e^{-\mathrm{i}\alpha},\ -\frac{w}{2}\right),$$

$$0 \le \alpha < 2\pi, \quad w \in \mathbb{C}, \quad \delta \in \mathbb{R}.$$

If w is represented in the form $2re^{\mathrm{i}\theta}$, then the invariant measure ds on S_4 can be written as

$$ds \equiv ds(2re^{\mathrm{i}\theta},\alpha,\delta) = r\,dr\,d\theta\,d\alpha\,d\delta. \tag{15}$$

The subgroup of real matrices $g(a,b,c,d)$ with $c>0$ is denoted by S_1. The matrices

$$g(\tau,r,s) = \begin{pmatrix} e^{2\tau} & 0 & r \\ 0 & e^{\tau} & s \\ 0 & 0 & 1 \end{pmatrix}, \quad r,s \in \mathbb{R}, \quad \tau > 0,$$

also form a group. It is denoted by S_2. The group of the matrices

$$g(\mathrm{i}\psi,r,s) = \begin{pmatrix} e^{2\mathrm{i}\psi} & 0 & r \\ 0 & e^{\mathrm{i}\psi} & s \\ 0 & 0 & 1 \end{pmatrix}, \quad r,s \in \mathbb{R}, \quad 0 \le \psi < 2\pi,$$

is denoted by S_3.

The matrices $g(a,b,1,d)$ from S form the three-dimensional *Heisenberg group* H, which is of great importance for physics.

§2. Construction of Representations

2.1. The Nonunitary Spherical Series of Representations. Let G, K, A, N, M, P be as in Sect. 1.1. We choose a one–dimensional representation

$$\lambda(h) = \exp \nu(H), \quad h \in \exp H,$$

of the subgroup $A = \exp \mathfrak{a}$. Then the correspondence

$$p \equiv mhn \to \delta(mhn) \equiv \lambda(h) \tag{16}$$

defines a one–dimensional representation of the minimal parabolic subgroup $P = MAN$. It induces the representation T_ν of the group G which acts in the space of functions $f(g)$ on G satisfying the condition

$$f(gp) = \lambda^{-1}(h)f(g), \quad p \equiv mhn \in P. \tag{17}$$

The operators $T_\nu(g_0)$, $g_0 \in G$, act on these functions by the formula

$$T_\nu(g_0)f(g) = f(g_0^{-1}g). \tag{18}$$

The representations T_ν belong to the nonunitary spherical series of representations of the group G. The representations T_ν are unitary if $\nu+\rho$ is pure imaginary on $\mathfrak{a}$. Recall that ρ is defined by formula (2a).

If functions f on G satisfy the condition (17), then they are determined by their values on certain subgroups of G. The Iwasawa decomposition $G = KAN$ shows that they are determined by their values on K and the relation $f(km) = f(k)$, $m \in M$, is satisfied. The operators $T_\nu(g)$ are given on $f(k)$, $k \in K$, by the formula

$$T_\nu(g)f(k) = \lambda(h^{-1})f(k_g), \tag{18a}$$

where $h \in A$ and $k_g \in K$ are defined by the Iwasawa decomposition $g^{-1}k = k_g hn$ of the element $g^{-1}k$. The scalar product

$$(f_1,f_2) = \int_K f_1(k)\overline{f_2(k)}\,dk$$

is introduced in the space of the functions f. The relation $f(km) = f(k)$, $m \in M$, means that the functions f actually are functions on the quotient space K/M. Thus, the representations T_ν are realized in the space of functions on K/M.

The Gauss decomposition $G = \overline{N}MAN$ shows that the functions f from (17) can be defined by their values on the subgroup $\overline{N}$. In this case the operators $T_\nu(g)$ take the form

$$T_\nu(g)f(n) = \lambda(h^{-1})f(n_g), \quad n \in \overline{N}, \tag{19}$$

where $h \in A$ and $n_g \in \overline{N}$ are determined by the decomposition $g^{-1}n = n_g mhn'$, $m \in M$, $n' \in N$.

To obtain a more general class of representations of G we have to replace the representations (16) of the subgroup P by the representations

$$p = mhn \to \delta(mhn) \equiv \omega(m)\lambda(h), \tag{20}$$

where ω is a unitary irreducible representation of the subgroup M, and to induce from them to the representations $T_{\omega\lambda}$ of the so-called nonunitary principal series.

The restriction $T_{\omega\lambda}\downarrow_K$ of the representation $T_{\omega\lambda}$ of the group G onto K is reducible. The multiplicity of an irreducible representation δ of the subgroup K in $T_{\omega\lambda}\downarrow_K$ is equal to the multiplicity of the representation ω of the subgroup M in $\delta\downarrow_M$. We conclude from this assertion that in the set $\{T_{\omega\lambda}\}$ the representations T_ν, and only they, are of class 1 with respect to the subgroup K, that is, they contain with multiplicity one the identity (trivial) representation of this subgroup.

2.2. Representations of the Group $SL(2,\mathbb{R})$. The subgroup M of $SL(2,\mathbb{R})$ consists of two elements $\pm e$ where e is the unit matrix. Therefore, the representations (20) of the parabolic subgroup $P = MAN$ of the group $SL(2,\mathbb{R})$ are given by two numbers $\chi = (\tau,\varepsilon)$, $\tau \in \mathbb{C}$, $\varepsilon \in \{0,1/2\}$ ($\varepsilon = 1/2$ corresponds to the nontrivial representation of M). Let us realize the representations $T_\chi \equiv T_{(\tau,\varepsilon)}$ of the nonunitary principal series of $SL(2,\mathbb{R})$ in the space of functions on the subgroup N, that is, in the space of functions $f(x)$ of a real variable. From formula (19), for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})$ we have

$$T_\chi(g)f(x) = |\beta x + \delta|^{2\tau}\operatorname{sign}^{2\varepsilon}(\beta x + \delta) f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right). \tag{20a}$$

If $\tau = i\rho - 1/2$, $\rho \in \mathbb{R}$, then the representations T_χ are unitary with respect to the scalar product of the Hilbert space $L^2(\mathbb{R})$. They constitute the unitary principal sèries (see Vilenkin [1965b], Chap. 6, Sect. 2.7).

The group $SL(2,\mathbb{R})$ also has a discrete series of unitary representations. The negative discrete series representations T_l^-, $l = -1, -3/2, -2, -5/2, \ldots$, act in the Hilbert spaces H_l of functions which are analytic in the upper half-plane $\mathbb{C}_+$. The scalar product in H_l is

$$(F_1, F_2) = \frac{i}{2\Gamma(-2l-1)} \int_{\mathbb{C}_+} F_1(w)\overline{F_2(w)}\, y^{-2l-2} dw\, d\overline{w}, \tag{21}$$

where $w = x + iy$ and $dw\, d\overline{w} = -2i\, dx\, dy$. The operators $T_l^-(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, are defined by the formula

$$T_l^-(g)F(w) = (\beta w + \delta)^{2l} F\left(\frac{\alpha w + \gamma}{\beta w + \delta}\right). \tag{22}$$

The positive discrete series representations T_l^+, $l = 1, 3/2, 2, \ldots$, are constructed in the same way in the Hilbert space of functions which are analytic in the lower half-plane.

Since $SU(1,1) \sim SL(2,\mathbb{R})$, the nonunitary principal series representations of the group $SU(1,1)$ are given by the same pair of numbers $\chi = (\tau,\varepsilon)$ as in the case of the group $SL(2,\mathbb{R})$. Realizing these representations on the subgroup $K = SO(2)$, we have

$$T_\chi(g)f(e^{i\theta}) = (be^{i\theta} + \overline{a})^{\tau+\varepsilon}(\overline{b}e^{-i\theta} + a)^{\tau-\varepsilon} f\left(\frac{ae^{i\theta} + \overline{b}}{be^{i\theta} + \overline{a}}\right), \tag{23}$$

where $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in SU(1,1)$.

2.3. Representations of the Group $SO_0(n,1)$. For the group $SO_0(n,1)$ we have $\mathfrak{a} = \{t(E_{n,n+1} + E_{n+1,n}) |\ t \in \mathbb{R}\}$ where E_{ij} is the matrix with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. Therefore, the characters of the subgroup $A = \exp \mathfrak{a}$ and, consequently, the representations T_ν of the spherical nonunitary series of the group $SO_0(n,1)$ are given by one complex number σ. For this reason we denote them by $T^\sigma, \sigma \in \mathbb{C}$.

Since for $SO_0(n,1)$ we have $K = SO(n)$ and $M = SO(n-1)$, the representations T^σ are realized in the space $L^2(S^{n-1})$ of functions f on the sphere $S^{n-1} = SO(n)/SO(n-1)$ of $\mathbb{R}^n$. These functions f can be considered as functions of the spherical coordinates $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ on S^{n-1}. Using assertions of Sect.1.3 we find that for the element $g_n'(t) \in A$ the operator $T_\sigma(g_n'(t))$ is of the form

$$\begin{aligned} T_\sigma(g_n'(t))f(\varphi_1, \ldots, \varphi_{n-1}) \\ = (\cosh t - \cos\varphi_{n-1}\sinh t)^\sigma f(\varphi_1, \ldots, \varphi_{n-2}, \varphi_{n-1}'), \end{aligned} \tag{23'}$$

where

$$\cos\varphi_{n-1}' = \frac{\cos\varphi_{n-1}\cosh t - \sinh t}{\cosh t - \cos\varphi_{n-1}\sinh t}.$$

For the elements $k \in SO(n)$ we have $T_\sigma(k)f(\boldsymbol{\xi}) = f(k^{-1}\boldsymbol{\xi})$, $\boldsymbol{\xi} \in S^{n-1}$.

The nonunitary principal series representations $T_{\omega\sigma}$ of the group $SO_0(n,1)$ are given by a complex number σ and by an irreducible unitary representation ω of the subgroup $M = SO(n-1)$.

2.4. Finite-Dimensional Representations. A description of finite-dimensional irreducible representations of compact groups can be found in the paper by Kirillov [1988]. For the theory of special functions it is useful to obtain them from the representations $T_{\omega\lambda}$ of the nonunitary principal series of the corresponding noncompact groups. This can be done in the following way.

Let G and G_k be dual noncompact and compact real semisimple Lie groups, that is, groups with the same complexification G_c. The groups G and G_k

have the same finite-dimensional representations which are restrictions to G and G_k of complex-analytic finite-dimensional representations of the group G_c (Zhelobenko [1970]). Since $G = KAK$ and $G_k = KA_kK$, $A = \exp \mathfrak{a}$, $A_k = \exp i\mathfrak{a}$, then finite-dimensional representations for G_k are obtained from those for G with the help of analytic continuation $A \to A_k$ (in terms of parameters) of matrix functions $T(g)$ of finite-dimensional representations T of the group G_c.

Every finite-dimensional irreducible representation T of the group G is contained as a subrepresentation of some representation $T_{\omega\lambda}$ of the nonunitary principal series of this group. Moreover, T is of class 1 with respect to K if and only if $T_{\omega\lambda}$ has this property.

Let H be the subspace of the carrier Hilbert space L of the representation $T_{\omega\lambda}$, in which the subrepresentation T is realized. Then after analytic continuation $A \to A_k$ we obtain the representation T of G_k, realized in the space H. As a rule, this representation of G_k is not unitary with respect to the inner product of the space L. To unitarize it we have to equip H with a new scalar product.

2.5. Representations of the Group $SU(2)$. Let the representation (23) of the group $SU(1,1)$ be such that $\tau+\varepsilon$ and $\tau-\varepsilon$ are integers and $\tau \geq 0$. The formula (23) shows that in this case the subspace, spanned by the functions $e^{in\theta}$, $n = -\tau+\varepsilon+j$, $j = 0,1,\ldots,2\tau$, is invariant, that is, a finite-dimensional representation of the group $SU(1,1)$ is realized in it. Let us multiply these functions by $e^{(\tau-\varepsilon)\theta}$, replace $e^{in\theta}$ by x and τ by l ($l = 0, 1/2, 1, 3/2, \ldots$), and then go over from the representation of $SU(1,1)$ to the corresponding representation of the dual group $SU(2)$. As a result, we obtain the representation T_l of the group $SU(2)$, realized in the space H_l of polynomials of degree less than or equal to $2l$ in one variable. It is given by the formula

$$T_l(u)f(x) = (\beta x + \overline{\alpha})^{2l} f\left(\frac{\alpha x - \overline{\beta}}{\beta x + \overline{\alpha}}\right), \quad u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2). \tag{24}$$

Evaluating the scalar product in H_l, for which the operators (24) are unitary (see Vilenkin [1965b], Chap. 3), we find that the functions

$$\psi_k(x) = \frac{(-ix)^{l-k}}{\sqrt{(l-k)!(l+k)!}}, \quad k = -l, -l+1, \ldots, l, \tag{25}$$

constitute an orthonormal basis of the space H_l. It is called the *canonical basis*.

2.6. Representations of the Group $SO(n+1)$. With the help of more complicated procedures, finite-dimensional representations T^l, $l = 0,1,2,\ldots$, of the group $SO(n+1)$ can be constructed by making use of the representations T^σ of $SO_0(n,1)$. The representations T^l of $SO(n+1)$ act in the spaces $\mathfrak{H}^{nl}$ of homogeneous harmonic polynomials in $x_1,\ldots,x_n$ of power $\leq l$. Such polynomials are uniquely defined by their values on the sphere S^{n-1}. As a result, we obtain from $\mathfrak{H}^{nl}$ the space $\mathfrak{D}^{nl}$ of functions $f(\varphi_1,\ldots,\varphi_{n-1})$ on S^{n-1} where $\varphi_1,\ldots,\varphi_{n-1}$ are spherical coordinates. The operators $T^l(k)$, $k \in SO(n)$, are given by the formula $T^l(k)f(\boldsymbol{\xi}) = f(k\boldsymbol{\xi})$ and the operators $T^l(g_n(\theta))$ by the formula

$$\begin{aligned} T^l(g_n(\theta))f(\varphi_1,\ldots,\varphi_{n-1}) \\ = (\cos\theta - i\sin\varphi_{n-1}\sin\theta)^l f(\varphi_1,\ldots,\varphi_{n-2},\varphi'_{n-1}), \end{aligned} \tag{26}$$

where

$$\cos\varphi'_{n-1} = \frac{\cos\varphi_{n-1}\cos\theta - i\sin\theta}{\cos\theta - i\cos\varphi_{n-1}\sin\theta}.$$

2.7. Representations of Inhomogeneous Groups. With every representation T_ν of the group G from Sect. 2.1 we associate a representation Q_ν of the corresponding inhomogeneous group G_s. It acts in the same space of functions on K, and is defined by the formula

$$Q_\nu(g)f(k) = \exp(-\nu(h))f(k_g), \tag{27}$$

where $h \in \mathfrak{a}$ and $k_g \in K$ are determined from the decomposition

$$g^{-1}(k,0) = (k_g,0)(e,t)(e,h), \quad t \in \mathfrak{t}$$

(see formula (12a)). The representations $Q_{\omega\lambda}$ of the group G_s, corresponding to the representations $T_{\omega\lambda}$ of G, are analogously constructed.

Let $G_s = ISO(2)$ be the group consisting of the matrices

$$g(\varphi;a,b) = \begin{pmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b \in \mathbb{R}, \quad 0 \leq \varphi < 2\pi.$$

It corresponds to the simple Lie group $G = SO_0(2,1) \sim SU(1,1)$. The representations Q_σ, $\sigma \in \mathbb{C}$, of $ISO(2)$ act on the space $L^2(0,2\pi)$ and are given by the formula

$$Q_\sigma(g)f(\psi) = e^{\sigma r\cos(\psi-\alpha)}f(\psi-\varphi), \quad g \equiv g(\varphi;a,b), \tag{28}$$

where $a = r\cos\alpha$ and $b = r\sin\alpha$.

Let $G_s = ISO(n)$ be the group of the matrices

$$g(k,\mathbf{a}) = \begin{pmatrix} k & \mathbf{a} \\ 0 & 1 \end{pmatrix}, \quad k \in SO(n), \quad \mathbf{a}^t = (a_1,\ldots,a_n), \quad a_j \in \mathbb{R}.$$

It corresponds to the simple group $G = SO_0(n,1)$. The representations Q_σ, $\sigma \in \mathbb{C}$, of the group $ISO(n)$ act in the space $L^2(S^{n-1})$. If $g_r = g(e,\mathbf{r})$, where e is the identity element of $SO(n)$ and $\mathbf{r} = (0,\ldots,0,r)$, then

$$Q_\sigma(g_\tau)f(\varphi_1,\ldots,\varphi_{n-1}) = e^{\sigma\tau\cos\varphi_{n-1}}f(\varphi_1,\ldots,\varphi_{n-1}). \tag{29}$$

The operators $Q_\sigma(k)$, $k \in SO(n)$, act in $L^2(S^{n-1})$ as left shifts by k^{-1}.

2.8. Representations of the Group of Transformations of the Straight Line. Let $G = I\mathbb{R}_+$ be the group of the transformations $x \to g(a,b)x \equiv ax + b$, $a > 0$, $-\infty < b < \infty$, of the straight line. It is the semidirect product of the group $\mathbb{R}_+$ of positive numbers with the group of shifts of the straight line. Let $\mathfrak{D}$ be the space of infinitely differentiable finite functions on the half line $0 < x < \infty$, vanishing in some neighborhood of the point $x = 0$. We fix the number λ and construct the operators

$$R_\lambda(g(a,b))\varphi(x) = e^{\lambda bx}\varphi(ax), \quad g \equiv g(a,b) \in I\mathbb{R}_+,$$

in $\mathfrak{D}$. The correspondence $g \to R_\lambda(g)$ gives a representation of the group $I\mathbb{R}_+$.

There is another realization of the representations R_λ which is more convenient in the theory of special functions. To construct it we go over from the functions $\varphi(x)$ to the functions

$$F(w) = \int_0^\infty \varphi(x)x^{w-1}dx. \tag{30}$$

Then

$$R_\lambda(g)F(w) = \int_0^\infty e^{\lambda bx}\varphi(xa)x^{w-1}dx = a^{-w}\int_0^\infty e^{\lambda bx/a}\varphi(x)x^{w-1}dx. \tag{31}$$

If $b = 0$, then we obtain

$$R_\lambda(g)F(w) = e^{-w}F(w). \tag{32}$$

Consequently, the operators $R_\lambda(g)$, corresponding to the one-parameter subgroup $\{g(a,0)\}$, are diagonal for this realization.

2.9. Representations of the Group $ISO(1,1)$. Elements of the group $ISO(1,1)$ are of the form

$$g \equiv g(\varphi; a, b) = \begin{pmatrix} \cosh\varphi & \sinh\varphi & a_1 \\ \sinh\varphi & \cosh\varphi & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We fix a complex number R and for every element $g \in ISO(1,1)$ construct the operator

$$T_R(g)\Phi(\theta) = e^{R(-a_1\cosh\theta + a_2\sinh\theta)}\Phi(\theta - \varphi)$$

in the space of infinitely differentiable finite functions on the hyperbola $\cosh^2\theta - \sinh^2\theta = 1$. The correspondence $g \to T_R(g)$ is a representation of the group $ISO(1,1)$.

Let us pass from the functions $\Phi(\theta)$ to their Fourier transforms

$$F(\lambda) = \int_{-\infty}^\infty \Phi(\theta)e^{\lambda\theta}d\theta. \tag{33}$$

Since the functions $\Phi(\theta)$ are finite, then

$$\Phi(\theta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\mu)e^{-\mu\theta}d\mu. \tag{34}$$

The operators $T_R(g)$ are transfered by the Fourier transform into the operators $Q_R(g)$. For the elements $g = g(\varphi; 0, 0)$ we have

$$Q_R(g)F(\lambda) = e^{\varphi\lambda}F(\lambda). \tag{35}$$

Consequently, if the representations T_R are realized on the functions $F(\lambda)$, then the operators corresponding to elements of the one-parameter subgroup $\{g(\varphi; 0, 0)\}$ are diagonal.

2.10. Representations of the Groups S and S_j. Let $\mathfrak{D}$ be the space of infinitely differentiable finite functions on the real line. For every pair $\chi = (\sigma, \omega)$ of complex numbers there is a representation T_χ of the group S_1 on the space $\mathfrak{D}$ given by the formula

$$T_\chi(g)f(x) = c^\omega e^{\sigma(dx+b)}f(cx + a). \tag{36}$$

Let us go over to the other realization of this representation. To construct it we associate with every function $f \in \mathfrak{D}$ the pair $F_+(\lambda)$, $F_-(\lambda)$ where

$$F_\gamma(\lambda) = \int_0^\infty x^{\lambda-1}f(\gamma x)\,dx \equiv \int_{-\infty}^\infty f(x)x_\gamma^{\lambda-1}dx; \quad \gamma = +,-; \quad \mathrm{Re}\,\lambda > 0. \tag{37}$$

Then

$$f(x) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} F_\gamma(\mu)|x|^{-\mu}d\mu, \quad \gamma = \mathrm{sign}\, x, \quad \rho > 0. \tag{38}$$

One can directly verify that for the matrices

$$\varepsilon(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$T_\chi(\varepsilon(\tau))\mathbf{F}(\lambda) = e^{(\omega-\lambda)\tau}\mathbf{F}(\lambda), \quad T_\chi(z(t))\mathbf{F}(\lambda) = e^{\sigma t}\mathbf{F}(\lambda), \tag{39}$$

where $\mathbf{F}(\lambda) = (F_+(\lambda), F_-(\lambda))$. Consequently, the operators corresponding to the one-parameter subgroups $\varepsilon(\tau)$ and $z(t)$ are diagonal for this realization.

Some of representations of the group S can be obtained by analytic continuation from the representations T_χ of the group S_1. These representations of S are given by the formula (36) and act on the space $\mathfrak{T}$ of entire analytic functions of exponential growth.

The representations T_α, $\alpha \in \mathbb{C}$, of the group S_2 act in the space $\mathfrak{D}_+$ of functions, given on $(0,\infty)$, and are defined by the formula

$$T_\alpha(g)f(x) = e^{-\alpha(rx^2+sx)}f(e^\tau x), \tag{40}$$

where $g = g(\tau, r, s)$. If $\alpha \neq 0$, then the representation T_α is irreducible. For pure imaginary α the representation T_α is unitary with respect to the scalar product

$$(f_1, f_2) = \int_0^\infty f_1(x)\overline{f_2(x)}\,\frac{dx}{x}.$$

If we put

$$F(\lambda) = \int_0^\infty f(x)\,x^{\lambda-1}dx,$$

then the representation T_α is transformed into the equivalent representation Q_α. For $g = g(\tau, 0, 0)$ the operator $T_\alpha(g)$ is transformed into the operator

$$Q_\alpha(g)F(\lambda) = e^{-\lambda\tau}F(\lambda) \tag{41}$$

and for $g = g(0, r, s)$, $r \neq 0$, $\mathrm{Re}\,\alpha r > 0$, into the operator

$$Q_\alpha(g)F(\lambda) = \int_{a-i\infty}^{a+i\infty} K_\alpha(\lambda-\mu; r, s)F(\mu)d\mu, \quad \mathrm{Re}\,\lambda > \mathrm{Re}\,\mu, \tag{42}$$

where

$$K_\alpha(\lambda; r, s) = \frac{1}{2\pi i}\int_0^\infty x^{\lambda-1}e^{-\alpha(rx^2+sx)}dx, \quad \mathrm{Re}\,\lambda > 0. \tag{43}$$

The equality

$$T_\alpha(g)f(z) = e^{-\alpha(rz^2+sz)}f(e^{i\psi}z), \quad |z| = 1, \tag{44}$$

defines the representation of the group S_3 in the space $L^2(0, 2\pi)$, which is irreducible for $\alpha \neq 0$.

The group S_4 is a subgroup of S. Therefore, the representations T_χ, $\chi = (\sigma, \omega)$, of the group S in the space $\mathfrak{T}$ give the representations of S_4. To obtain unitary representations of S_4 we introduce on the space of entire analytic functions $f(z)$ on $\mathbb{C}$ the scalar product

$$(f_1, f_2) = \frac{1}{\pi}\int_{\mathbb{C}} f_1(z)\overline{f_2(z)}\exp(-|z|^2)dx\,dy. \tag{45}$$

As a result, we receive the Hilbert space $\mathfrak{H}$, in which the representations $T_{(\rho,m)}$, $\rho > 0$, $m \in \mathbb{Z}$, are unitary, where $\mathbb{Z}$ is the set of integers (Miller [1968a]).

Chapter 2
Matrix Elements of Representations and Special Functions

§1. Matrix Elements of Group Representations

1.1. Properties of Matrix Elements. Let T^α be a representation of a group G in a Hilbert space $\mathcal{H}$. The scalar-valued function $t^\alpha_{\mathbf{xy}}(g) = (T^\alpha(g)\mathbf{y}, \mathbf{x})$ corresponds to the pair of vectors $\mathbf{x}$ and $\mathbf{y}$ from $\mathcal{H}$. It is called the *matrix element* of the representation T^α. If $\{\mathbf{e}_i\}$ is an orthonormal basis of $\mathcal{H}$, then the matrix element $t^\alpha_{\mathbf{e}_i\mathbf{e}_j}(g)$ is denoted by $t^\alpha_{ij}(g)$.

The formula $T^\alpha(g_1g_2) = T^\alpha(g_1)T^\alpha(g_2)$ implies the equality

$$t^\alpha_{ij}(g_1g_2) = \sum_k t^\alpha_{ik}(g_1)t^\alpha_{kj}(g_2). \tag{1}$$

Let $g(u)$ be a one-parameter subgroup. Setting $t^\alpha_{ij}(g(u)) = t^\alpha_{ij}(u)$ we obtain from (1) that

$$t^\alpha_{ij}(u+v) = \sum_k t^\alpha_{ik}(u)t^\alpha_{kj}(v). \tag{2}$$

Differentiating this relation in u and putting $u = 0$ we have

$$\frac{d}{dv}t^\alpha_{ij}(v) = \sum_k b^\alpha_{ik}t^\alpha_{kj}(v), \tag{3}$$

where $b^\alpha_{ik} = \frac{d}{du}\left(t^\alpha_{ik}(u)\right)\big|_{u=0}$. We analogously derive that

$$\frac{d}{du}t^\alpha_{ij}(u) = \sum_k t^\alpha_{ik}(u)b^\alpha_{kj}. \tag{4}$$

Thus, the functions $t^\alpha_{ij}(u)$, $1 \le i \le \dim T^\alpha$, with fixed j are solutions of the system of differential equations (3), satisfying the initial condition $t^\alpha_{ij}(0) = \delta_{ij}$. Similarly, the functions $t^\alpha_{ij}(u)$, $1 \le j \le \dim T^\alpha$, with fixed i are solutions of the system (4) with the initial condition $t^\alpha_{ij}(0) = \delta_{ij}$.

Matrix elements of unitary representations satisfy the relation

$$\sum_k t^\alpha_{ik}(g)\overline{t^\alpha_{jk}(g)} = \delta_{ij}. \tag{5}$$

We deduce from this that $|t^\alpha_{ij}(g)| \le 1$.

If $\mathbf{e}_i$ is a basis element of the carrier space of the unitary representation T^α of G, then the matrix element $t^\alpha_{ii}(g)$ is a positive definite function; that is, for every finite set of elements $g_1, \ldots, g_n$ from G and for every choice of complex numbers $c_1, \ldots, c_n$ the relation