6. SUPER SPACETIMES AND SUPER POINCARÉ GROUPS

6.1. Super Lie groups and their super Lie algebras. The definition of a super Lie group within the category of supermanifolds imitates the definition of Lie groups within the category of classical manifolds. A *real super Lie group* G is a real supermanifold with morphisms

$$m : G \times G \longrightarrow G, \qquad i : G \longrightarrow G$$

which are multiplication and inverse, and

$$1 : \mathbf{R}^{0|0} \longrightarrow G$$

defining the unit element, such that the usual group axioms are satisfied. However in formulating the axioms we must take care to express then entirely in terms of the maps $m, i, 1$. To formulate the associativity law in a group, namely, $a(bc) = (ab)c$, we observe that $a, b, c \longmapsto (ab)c$ may be viewed as the map $I \times m : a, (b, c) \longmapsto a, bc$ of $G \times (G \times G) \longrightarrow G \times G$ (I is the identity map), followed by the map $m : x, y \longmapsto xy$. Similarly one can view $a, b, c \longmapsto (ab)c$ as $m \times I$ followed by m. Thus the associativity law becomes the relation

$$m \circ (I \times m) = m \circ (m \times I)$$

between the two maps from $G \times G \times G$ to G. We leave it to the reader to formulate the properties of the inverse and the identity. The identity of G is a point of G_{red}. It follows almost immediately from this that if G is a super Lie group, then G_{red}

is a Lie group in the classical sense. Also we have defined real super Lie groups above without specifying the smoothness type. One can define smooth or analytic Lie groups by simply taking the objects and maps to be those in the category of analytic super manifolds. Similarly for complex super Lie groups.

The functor of points associated to a super Lie group reveals the true character of a super Lie group. Let G be a super Lie group. For any supermanifold S let $G(S)$ be the set of morphisms from S to G. The maps $m, i, 1$ then give rise to maps

$$m_S : G(S) \times G(S) \longrightarrow G(S), \qquad i_S : G(S) \longrightarrow G(S), \qquad 1_S : 1_S \longrightarrow G(S)$$

such that the group axioms are satisfied. This means that the functor

$$S \longmapsto G(S)$$

takes values in groups. Moreover, if T is another supermanifold and we have a map $S \longrightarrow T$, the corresponding map $G(T) \longrightarrow G(S)$ is a homomorphism of groups. Thus $S \longmapsto G(S)$ is a *group-valued functor*. One can also therefore define a super Lie group as a functor

$$S \longmapsto G(S)$$

from the category of supermanifolds to the category of groups which is representable by a supermanifold G. The maps $m_S : G(S) \times G(S) \longrightarrow G(S)$, $i_S : G(S) \longrightarrow G(S)$, and 1_S then define, by Yoneda's lemma, maps $m, i, 1$ that convert G into a super Lie group and $S \longmapsto G(S)$ is the functor of points corresponding to G. A morphism of super Lie groups $G \longrightarrow H$ is now one that commutes with $m, i, 1$. It corresponds to homomorphisms

$$G(S) \longrightarrow H(S)$$

that are functorial in S. If G and H are already defined, Yoneda's lemma assures us that morphisms $G \longrightarrow H$ correspond one-one to homomorphisms $G(S) \longrightarrow H(S)$ that are functorial in S.

The actions of super Lie groups on supermanifolds are defined exactly in the same way. Thus if G is a super Lie group and M is a supermanifold, actions are defined either as morphisms $G \times M \longrightarrow M$ with appropriate axioms or as actions $G(S) \times M(S) \longrightarrow M(S)$ that are functorial in S; again Yoneda's lemma makes such actions functorial in S to be in canonical bijection with actions $G \times M \longrightarrow M$.

Sub super Lie groups are defined exactly as in the classical theory. A super Lie group H is a subgroup of a super Lie group if H_{red} is a Lie subgroup of G_{red} and the inclusion map of H into G is a morphism which is an immersion everywhere. One of the most usual ways of encountering sub super Lie groups is as stabilizers of points in actions. Suppose that G acts on M and m is a point of M_{red}. Then, for any supermanifold S, we have the stabilizer $H(S)$ of the action of $G(S)$ on $M(S)$ at the point m_S of $M(S)$. The assignment $S \longmapsto H(S)$ is clearly functorial in S. It can the be shown that this functor is representable, and that the super Lie group it defines is a closed sub super Lie group of G.

As in the classical case products of super Lie groups are super Lie groups. The opposite of a super Lie group is also one.

The super Lie algebra of a super Lie group. In the classical theory the Lie algebra of a Lie group is defined as the Lie algebra of left (or right) invariant vector fields on the group manifold with the bracket as the usual bracket of vector fields. The left invariance guarantees that the vector field is uniquely determined by the tangent vector at the identity; one starts with a given tangent vector at the identity and then translates it to each point to obtain the vector field. In the case of a super Lie group we follow the same procedure, but much more care is required because we have to consider not only the topological points but others also. The main point is that if M is a supermanifold and v is a vector field on M - for instance defined in local coordinates as $\sum_i a_i\partial/\partial x^i + \sum_j b_j\partial/\partial\theta^j$ where a_i, b_j are sections of the structure sheaf locally, then v is *not* determined by the tangent vectors it defines at each point.

For a super Lie group it is now a question of making precise what is a left invariant vector field. If we are dealing with a classical Lie group G, the left invariance of a vector field X is the relation $\ell_x \circ X = X \circ \ell_x$ for all $x \in G$ where ℓ_x is left translation by x, i.e.,

$$X_y f(xy) = (Xf)(xy)$$

for all $x, y \in G$ where X_y means that X acts only on the second variable y. This can also be written as

$$(I \otimes X) \circ m^* = m^* \circ X \tag{1}$$

where m^* is the sheaf morphism from $\mathcal{O}_G$ to $\mathcal{O}_{G\times G}$ corresponding to the multiplication $m : G \times G \longrightarrow G$. Now this definition can be taken over to the super case without change. The following is the basic theorem.

Theorem 6.1.1. *The Lie algebra* $\mathfrak{g}$ *of a super Lie group* G *is the set of all vector fields* X *on* G *satisfying* (1). *It is a super Lie algebra of the same dimension as* G. *The map* $X \longmapsto X_1$ *that sends* $X \in \mathfrak{g}$ *to the tangent vector at the identity point* 1 *is a linear isomorphism of super vector spaces. If* τ *is a tangent vector to* G *at* 1, *the vector field* $X \in \mathfrak{g}$ *such that* $X_1 = \tau$ *is the unique one such that*

$$Xf = (I \otimes \tau)(m^*(f)) \tag{2}$$

for all (local) sections of $\mathcal{O}_G$. Finally, the even part of $\mathfrak{g}$ is the Lie algebra of the classical Lie group underlying G, i.e.,

$$\mathfrak{g}_0 = \mathrm{Lie}(G_{\mathrm{red}}).$$

Remark. We shall not prove this here. Notice that equation (2) can be interpreted formally as

$$(Xf)(x) = (\tau_y)(f(xy)). \tag{3}$$

Thus the elements of the Lie algebra of G can be obtained by differentiating the group law exactly as we do classically. This will become clear in the examples we consider below. However as a simple example consider $G = \mathbf{R}^{1|1}$ with global coordinates x, θ. We introduce the group law

$$(x, \theta)(x', \theta') = (x + x' + \theta\theta', \theta + \theta')$$

with the inverse

$$(x, \theta)^{-1} = (-x, -\theta).$$

The Lie algebra is of dimension $1|1$. If D_x, D_θ are the left invariant vector fields that define the tangent vectors $\partial_x = \partial/\partial x, \partial_\theta = \partial/\partial\theta$ at the identity element 0, and D_x^r, D_θ^r are the corresponding right invariant vector fields, then the above recipe yields

$$\begin{aligned} D_x &= \partial_x, & D_x^r &= \partial_x \\ D_\theta &= \theta\partial_x + \partial_\theta, & D_\theta^r &= -\theta\partial_x + \partial_\theta. \end{aligned}$$

It is now an easy check that

$$[D_x, D_\theta] = 2D_x$$

(all other commutators are zero) giving the structure of the Lie algebra. A similar method yields the Lie algebras of $GL(p|q)$ and $SL(p|q)$; they are respectively $\mathfrak{gl}(p|q)$ and $\mathfrak{sl}(p|q)$.

Theorem 6.1.2. *For morphism $f : G \longrightarrow G'$ of super Lie groups G, G' we have its differential Df which is a morphism of the corresponding super Lie algebras, i.e., $Df : \mathfrak{g} \longrightarrow \mathfrak{g}'$. It is uniquely determined by the relation $Df(X)_{1'} = df_1(X_1)$ where $1, 1'$ are the identity elements of G, G' and df_1 is the tangent map $T_1(G) \longrightarrow T_{1'}(G')$. Moreover f_{red} is a morphism $G_{\mathrm{red}} \longrightarrow G'_{\mathrm{red}}$ of classical Lie groups.*

The fundamental theorems of Lie go over to the super category without change. All topological aspects are confined to the classical Lie groups underlying the super

Lie groups. Thus, a morphism $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}'$ comes from a morphism $G \longrightarrow G'$ if and only if $\alpha_0 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ comes from a morphism $G_{\text{red}} \longrightarrow G'_{\text{red}}$. The story is the same for the construction of a super Lie group corresponding to a given super Lie algebra: given a classical Lie group H with Lie algebra $\mathfrak{g}_0$, there is a unique super Lie group G with $\mathfrak{g}$ as its super Lie algebra such that $G_{\text{red}} = H$. The classification of super Lie algebras over $\mathbf{R}$ and $\mathbf{C}$ and their representation theory thus acquires a geometric significance that plays a vital role in supersymmetric physics.

Super affine algebraic groups. There is another way to discuss Lie theory in the supersymmetric context, namely as algebraic groups. In the classical theory algebraic groups are defined as groups of matrices satisfying polynomial equations. Examples are $GL(n), SL(n), SO(n), Sp(2n)$ and so on. They are affine algebraic varieties which carry a group structure such that the group operations are morphisms. If R is a commutative k-algebra with unit element, $G(S)$ is the set of solutions to the defining equations; thus we have $GL(n,R), SL(n,R), SO(n,R), Sp(2n,R)$. In general an affine algebraic group scheme defined over k is a functor $R \longmapsto G(R)$ from the category of commutative k-algebras with units to the category of groups which is representable. Representability means that there is a commutative algebra with unit, $k[G]$ say, such that

$$G(R) = \text{Hom}\ (k[G], R)$$

for all R. By Yoneda's lemma the algebra $k[G]$ acquires a coalgebra structure, an antipode, and a co unit, converting it into a *Hopf algebra.* The generalization to the super context is almost immediate: a super affine algebraic groups defined over k is a functor

$$R \longmapsto G(R)$$

from the category of supercommutative k-algebras to the category of groups which is representable, i.e., there is a supercommutative k-algebra with unit, $k[G]$ say, such that

$$G(R) = \text{Hom}\ (k[G], R)$$

for all R. The algebra $k[G]$ then acquires a super Hopf structure. The theory can be developed in parallel with the transcendental theory. Of course in order to go deeper into the theory we need to work with general super schemes, for instance when we deal with homogeneous spaces which are very often not affine but projective. The Borel subgroups and the super flag varieties are examples of these.

6.2. The Poincaré-Birkhoff-Witt theorem. The analog for super Lie algebras of the Poincaré-Birkhoff-Witt (PBW) theorem is straightforward to formulate. Let