

THE HOWE DUALITY AND POLYNOMIAL SOLUTIONS FOR THE SYMPLECTIC DIRAC OPERATOR

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ABSTRACT. We find the Fisher decomposition for the space of polynomials valued in the Segal-Shale-Weil representation. As a consequence, this allows to determine symplectic monogenics, i.e. the space of polynomial solutions of the symplectic Dirac operator.

1. INTRODUCTION

The classical topic of separation of variables, often realized by a Howe dual pair acting on the representation of interest, is one of the cornerstones in harmonic analysis.

Some well-known examples are the Howe dual pair for the orthogonal Lie group $O(n)$ acting on the space of polynomials on \mathbb{R}^n given by $O(n) \times \mathfrak{sl}_2$ and the dual pair for the representation on spinor valued polynomials given by $O(n) \times \mathfrak{osp}(1|2)$. Recently, there has been a lot of interest in deformations of these dual pairs where the orthogonal group is replaced by a finite reflection group $\mathcal{G} < O(n)$. This leads to the introduction of the so-called Dunkl operators (see e.g. [8, 9]). They allow to construct a Dunkl Laplacian and it was proven in [13] that the relevant operators again generate \mathfrak{sl}_2 (see also [1]). In [17], a Dunkl Dirac operator has been introduced, leading to a dual pair $\mathcal{G} \times \mathfrak{osp}(1|2)$.

The Dunkl Laplacian can be seen as a spherical deformation of the classical Laplacian. Very recently, also radial deformations have been obtained in this context. We refer the reader to [3] for radial deformations of \mathfrak{sl}_2 and to [6] for radial deformations of $\mathfrak{osp}(1|2)$.

A natural question to be posed now is whether similar things are also possible for the symplectic group $Sp(2n)$, namely do there exist dual pairs $Sp(2n) \times \mathfrak{sl}_2$ or $Sp(2n) \times \mathfrak{osp}(1|2)$ and do there exist Dunkl analogs of these dual pairs. In other words, can the operators realizing these dual pairs be deformed to be only invariant under a (finite) subgroup of the symplectic group?

The answer is partly positive. In e.g. [7, 5] a theory of harmonic analysis has been established for operators acting on Grassmann algebras (or superspaces), leading to dual pairs $Sp(2n) \times \mathfrak{sl}_2$ and $Sp(2n) \times \mathfrak{osp}(1|2)$. However, although this yields a satisfying framework, it is unclear how the operators obtained there can be deformed to be only invariant under a subgroup of the symplectic group. The

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construction leading to the classical Dunkl operators (see [8]) cannot immediately be transferred to this new framework, due to the different algebraic structure of Grassmann algebras.

Fortunately, there seems to be an alternative way of obtaining symplectic analogs of the orthogonal constructions. Many years ago B. Kostant introduced a symplectic analog of the Dirac operator called symplectic Dirac operator. This symplectic Dirac operator was mainly studied from the geometrical point of view, see [12] and references therein, and also as an invariant differential operator in [15], but its spectral properties are difficult to obtain.

In this article we aim to study the symplectic Dirac operator from the point of view of special functions and harmonic analysis. We will establish the Howe duality and the corresponding Fischer decomposition. As a consequence, we determine its kernel on \mathbb{R}^{2n} . It is important to note that the symplectic Dirac operator that we will study does not have a Laplace counterpart.

The paper is organized as follows. In section 2 and 3 we repeat some well-known facts on symplectic Lie algebras and their finite dimensional representations. In section 4 we define the symplectic Dirac operator and show how it leads to a realization of \mathfrak{sl}_2 . In section 5 we obtain the Fischer decomposition and construct explicit projection operators on all summands. We end with some conclusions and an outlook for further research.

2. SYMPLECTIC LIE ALGEBRA, SYMPLECTIC CLIFFORD ALGEBRA AND SIMPLE HIGHEST WEIGHT MODULES FOR $sp(2n)$

Let us consider a symplectic vector space $(\mathbb{R}^{2n}, \omega)$ and a symplectic basis

$$e_1, \dots, e_n, f_1, \dots, f_n$$

with respect to the non-degenerate two-form ω on \mathbb{R}^{2n} . Let $E_{i,j}$ be the $2n \times 2n$ matrix with 1 on the intersection of the i -th row and j -th column, and zero otherwise. The symplectic Lie algebra $sp(2n)$ is generated by (see e.g. [11])

$$X_{ij} = E_{i,j} - E_{n+i,n+j}, Y_{ij} = E_{i,n+j} + E_{j,n+i}, Z_{ij} = E_{n+i,j} + E_{n+j,i}$$

for $i, j \in \{1, \dots, n\}$, and can be realized by the first order differential operators

$$X_{ij} = x_j \partial_i - x_{n+j} \partial_{n+i}, Y_{ij} = x_{n+j} \partial_i + x_{n+i} \partial_j, Z_{ij} = x_j \partial_{n+i} + x_i \partial_{n+j}.$$

For example, in the case $n = 1$ the three vector fields $\{2x_2 \partial_1, 2x_1 \partial_2, x_1 \partial_1 - x_2 \partial_2\}$ generate $sp(2)$.

Next, we introduce the symplectic Clifford algebra over $(\mathbb{R}^{2n}, \omega)$ (see [4]). This is the associative algebra over \mathbb{R} with unit element, which is multiplicatively generated by the elements $e_1, \dots, e_n, f_1, \dots, f_n$ under the relations

$$\begin{aligned} e_i e_j &= e_j e_i \\ f_i f_j &= f_j f_i \\ e_i f_j - f_j e_i &= \omega(e_i, f_j) = \delta_{ij} \end{aligned}$$

for all $i, j \in \{1, \dots, n\}$.

The metaplectic Lie algebra $mp(2n, \mathbb{R})$ is a Lie algebra attached to the twofold covering $\rho : Mp(2n, \mathbb{R}) \rightarrow Sp(2n, \mathbb{R})$ of the symplectic Lie group $Sp(2n)$. It can be

realized by homogeneity two elements in the symplectic Clifford algebra, where the homomorphism $\rho_\star : mp(2n, \mathbb{R}) \rightarrow sp(2n, \mathbb{R})$ is given by

$$(2.1) \quad \begin{aligned} \rho_\star(e_i e_j) &= -Y_{ij}, \\ \rho_\star(f_i f_j) &= Z_{ij}, \\ \rho_\star(e_i f_j + f_j e_i) &= 2X_{ij} \end{aligned}$$

for $i, j \in \{1, \dots, n\}$. There is another useful realization of the symplectic Lie algebra as a subalgebra of the Weyl algebra of rank n . Let $x_i, i \in \{1, \dots, n\}$ be the generators of polynomial algebra. The Weyl algebra is the associative algebra generated by $\{x_i, \partial_i\}, i \in \{1, \dots, n\}$, partial differentiation with respect to x_i and multiplication operators x_i , acting on polynomials on \mathbb{R}^n . The root spaces of $sp(2n)$ corresponding to positive simple roots α_i are spanned by $x_{i+1} \partial_i, i \in \{1, \dots, n-1\}$ and α_n is spanned by $-\frac{1}{2} \partial_n^2$.

The Segal-Shale-Weil representation L is the minimal highest weight representation of $Sp(2n, \mathbb{R})$ on the vector space $L^2(\mathbb{R}^n, d\mu)$, where $d\mu = \exp^{-\|x\|^2} dx_{\mathbb{R}^n}$ with $dx_{\mathbb{R}^n}$ the Lebesgue measure on \mathbb{R}^n . We take for the basis of this vector space polynomials on the isotropic subspace $\mathbb{R}^n \subset \mathbb{R}^{2n}$. The differential $L_\star : mp(2n, \mathbb{R}) \rightarrow \text{End}(\text{Pol}(\mathbb{R}^n))$ of the Segal-Shale-Weil representation is

$$(2.2) \quad \begin{aligned} L_\star(e_i e_j) &= ix_i x_j, \\ L_\star(f_i f_j) &= -i \partial_i \partial_j, \\ L_\star(e_i f_j + f_j e_i) &= x_i \partial_j + x_j \partial_i \end{aligned}$$

for $i, j \in \{1, \dots, n\}$.

3. DECOMPOSITION OF TENSOR PRODUCTS OF FINITE DIMENSIONAL REPRESENTATIONS WITH THE SEGAL-SHALE-WEIL REPRESENTATION

In this section, we will make explicit several results in [2] on the decomposition of the tensor product of completely pointed highest weight modules (specifically, we will consider two irreducible components of the Segal-Shale-Weil representation) with a suitable class of finite dimensional representations (specifically, symmetric powers of the fundamental vector representation \mathbb{R}^{2n}) of $sp(2n, \mathbb{R})$. The property of a module being completely pointed means that all of its weight spaces are uniformly bounded by a constant, see [2], Lemma 2.1. Throughout the article, $V(\mu)$ denotes the Verma module of highest weight μ and $L(\mu)$ denotes the simple module of highest weight μ , i.e. the quotient of $V(\mu)$ by its unique maximal submodule $I(\mu) \subset V(\mu)$.

Let us introduce the set

$$\tau^i = \left\{ \sum_{j=1}^n d_j L_j \mid d_j + \delta_{1,i} \delta_{n,j} \in \mathbb{N}, \sum_{j=1}^n d_j = 0 \text{ mod } 2 \right\}.$$

Here \mathbb{N} means the set of natural numbers including 0, $d_j \in \mathbb{N}$ and $i = 0$ resp. $i = 1$ for $V(-\frac{1}{2}\omega_n)$ resp. $V(\omega_{n-1} - \frac{3}{2}\omega_n)$. These sets are bijective with the set of weights of two irreducible parts of the Segal-Shale-Weil representation.

Let $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ be a dominant integral weight, i.e. the highest weight of a finite dimensional irreducible representation. Define the set of weights

$$\begin{aligned} \tau_\lambda^i &= \{ \mu \mid \lambda - \mu = \sum_{j=1}^n d_j L_j \in \tau^i, \\ &0 \leq d_j \leq \lambda_j (j = 0, \dots, n-1), 0 \leq d_n + \delta_{1,i} \leq 2\lambda_n + 1 \}. \end{aligned}$$

The central result of [2] is

Theorem 3.1. *Let $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ denote the Verma modules corresponding to irreducible subrepresentations of the Segal-Shale-Weil representations. Then for any finite dimensional irreducible representation $F(\lambda)$ with highest weight λ we have*

$$(3.3) \quad L(-\frac{1}{2}\omega_n) \otimes F(\lambda) \simeq \bigoplus_{\mu \in \tau_\lambda^0} L(-\frac{1}{2}\omega_n + \mu)$$

(the same is true for $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ with τ_λ^1 instead of τ_λ^0 .) In particular, the decomposition is direct and so the tensor product is completely reducible.

Recall that the Verma modules $V(\mu)$ appearing in the previous decomposition are irreducible, i.e. equal to $L(\mu)$. In particular, they have no other singular vector than the highest weight one.

The consequence of this result is the decomposition of the tensor product of $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ with symmetric powers $S^k(\mathbb{C}^{2n})$ ($k \in \mathbb{N}$) of the fundamental vector representation \mathbb{C}^{2n} of $sp(2n)$. Note that these are irreducible representations (see [11]).

Corollary 3.2. *We have for $L(-\frac{1}{2}\omega_n)$*

(1) *In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side):*

$$\begin{aligned} L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) &\simeq L(-\frac{1}{2}\omega_n) \oplus L(\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \\ &\oplus L(2\omega_1 - \frac{1}{2}\omega_n) \oplus L(3\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \dots \\ &\oplus L((2l-1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(2l\omega_1 - \frac{1}{2}\omega_n), \end{aligned}$$

(2) *In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side):*

$$\begin{aligned} L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) &\simeq L(\omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(\omega_1 - \frac{1}{2}\omega_n) \\ &\oplus L(2\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(3\omega_1 - \frac{1}{2}\omega_n) \oplus \dots \\ &\oplus L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L((2l+1)\omega_1 - \frac{1}{2}\omega_n). \end{aligned}$$

We have for $L(\omega_{n-1} - \frac{3}{2}\omega_n)$

(1) *In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side):*

$$\begin{aligned} L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) &\simeq L(\omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(\omega_1 - \frac{1}{2}\omega_n) \\ &\oplus L(2\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \dots \\ &\oplus L((2l-1)\omega_1 - \frac{1}{2}\omega_n) \oplus L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n), \end{aligned}$$

(2) In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side):

$$\begin{aligned} L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) &\simeq L(-\frac{1}{2}\omega_n) \oplus L(\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \dots \\ &\oplus L(2l\omega_1 - \frac{1}{2}\omega_n) \oplus L((2l+1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n). \end{aligned}$$

Reformulated in the language of differential operators, this Corollary leads to Theorem 5.3. We also introduce the notation $\mathcal{C} = L(-\frac{1}{2}\omega_n) \oplus L(\omega_{n-1} - \frac{3}{2}\omega_n)$.

4. SYMPLECTIC STRUCTURE AND ITS LIE ALGEBRA \mathfrak{sl}_2

Let $(\mathbb{R}^{2n}, \omega)$ be the symplectic vector space with coordinates x_1, \dots, x_{2n} , coordinate vector fields $\partial_1, \dots, \partial_{2n}$ and symplectic frame $e_1, f_1, \dots, e_n, f_n$, i.e. $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$ for all $i, j = 1, \dots, n$. It follows from the action of $sp(2n)$ on these vectors that

$$\begin{aligned} X_s &:= \sum_{j=1}^n (x_{2j-1}f_j + x_{2j}e_j), \\ D_s &:= \sum_{j=1}^n (\partial_{x_{2j-1}}e_j - \partial_{x_{2j}}f_j), \\ \mathbb{E} &:= \sum_{j=1}^{2n} x_j \partial_{x_j} \end{aligned} \tag{4.4}$$

are invariant and so will be used as linear maps intertwining the $sp(2n)$ action on the space $\mathcal{P} \otimes \mathcal{C}$ of symplectic spinor valued polynomials on \mathbb{R}^{2n} , with $\mathcal{P} = Pol(\mathbb{R}^{2n})$. The space of homogeneous polynomials of degree k will be denoted by \mathcal{P}_k .

It is easy to verify that they fulfill \mathfrak{sl}_2 commutation relations:

$$\begin{aligned} [\mathbb{E} + n, D_s] &= -D_s, \\ [\mathbb{E} + n, X_s] &= X_s, \\ [D_s, X_s] &= \mathbb{E} + n. \end{aligned} \tag{4.5}$$

The action of $\mathfrak{sl}_2 \times sp(2n)$ will generate the multiplicity free decomposition of the representation of interest.

Further we introduce the operator

$$\Gamma_s = X_s D_s - \frac{1}{2} \mathbb{E}(2n - 1 + \mathbb{E}), \tag{4.6}$$

which is the Casimir operator in \mathfrak{sl}_2 . Using the formulas (4.5) it is easy to check that Γ_s commutes with both X_s and D_s .

5. FISCHER DECOMPOSITION AND HOMOMORPHISMS OF $sp(2n, \mathbb{R})$ -MODULES APPEARING IN THE DECOMPOSITION OF POLYNOMIALS VALUED IN THE SEGAL-SHALE-WEIL REPRESENTATION

Before introducing the scheme in full generality, we start with a few explicit remarks concerning homogeneity zero and one parts in the decomposition. The

tensor product $L(-\frac{1}{2}\omega_n) \otimes \mathbb{C}^{2n}$ (analogously, one can consider $L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes \mathbb{C}^{2n}$) decomposes as the direct sum $V_1 \oplus V_2$ of two invariant subspaces, given by

$$V_1 := \left\{ \sum_{i=1}^n e_i s \otimes e_i + \sum_{i=1}^n f_i s \otimes e_i \mid s \in L(-\frac{1}{2}\omega_n) \right\},$$

$$V_2 := \left\{ \sum_{i=1}^n s_i \otimes e_i + \sum_{j=1}^n s_j \otimes f_j; s_i, s_j \in L(-\frac{1}{2}\omega_n) \mid \sum_{i=1}^n e_i s_i + \sum_{j=1}^n f_j s_j = 0 \right\}.$$

The map $i : L(\omega_{n-1} - \frac{3}{2}\omega_n) \rightarrow L(-\frac{1}{2}\omega_n) \otimes \mathbb{C}^{2n}$ (resp. $L(-\frac{1}{2}\omega_n) \rightarrow L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes \mathbb{C}^{2n}$) is injective and onto V_1 . The reason is that injectivity $i(s) = 0$ is equivalent to $e_i s = 0$ resp. $f_i s = 0$ for all $i \in \{1, \dots, n\}$, and so $(e_i f_j - f_j e_i) s = 0$. The symplectic Clifford algebra relation $e_i f_j - f_j e_i = \delta_{ij}$ implies $s = 0$ and the result follows. In other words, the action of X_s induces an isomorphism between two irreducible submodules in homogeneity zero and one.

Another remark is an application of tools in representation theory, in particular of infinitesimal character. The sum of fundamental weights (or half of the sum of positive roots) for $sp(2n)$ is $\delta = (n, n-1, \dots, 2, 1)$. The highest weights of irreducible simple $sp(n)$ -modules, coming from the decomposition of the tensor product, were determined for each homogeneity $k \in \mathbb{N}$ in Corollary 3.2. The multiplication by X_s gives an intertwining map between neighboring columns, say the k -th and $(k+1)$ -th. Let us determine possible target modules when restricting the action of X_s to a given simple irreducible $sp(2n)$ -module $L(a\omega_1 - \frac{1}{2}\omega_n)$ with highest weight $a\omega_1 - \frac{1}{2}\omega_n$ for some $a \leq k$ (the case of $L(b\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n)$ being analogous.) The comparison of infinitesimal characters of the collection of weights $\{\mu_a = a\omega_1 - \frac{1}{2}\omega_n, \nu_b = b\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n\}$ ($a, b \in \mathbb{N}$) yields $\|\mu_a + \delta\|^2 = \|\nu_b + \delta\|^2$ if and only if either

- (1) $2a + n - \frac{1}{2} = 2b + n - \frac{1}{2}$, which implies $a = b$, or
- (2) $2a + n - \frac{1}{2} = -(2b + n - \frac{1}{2})$, i.e. $a + b = -n + \frac{1}{2}$ and there is no solution in this case.

It remains to prove that the image of X_s , when restricted to an irreducible simple module in the k -th column, is nonzero (or, as follows from irreducibility, is the irreducible simple module in the $(k+1)$ -th column with the same infinitesimal character.)

To complete this line of reasoning, we employ the Lie algebra \mathfrak{sl}_2 from Section 4. To illustrate it explicitly, we start in homogeneity zero with only one simple module $L(-\frac{1}{2}\omega_n)$ and assume that it is $\text{Ker}(X_s)$. Because it is in $\text{Ker}(D_s)$, it is in the kernel of the commutator $[D_s, X_s] = \mathbb{E} + n$. However, $\mathbb{E} + n$ acts in homogeneity zero by n , which is the required contradiction and so X_s acts as an isomorphism $L(-\frac{1}{2}\omega_n) \rightarrow L(\omega_{n-1} - \frac{3}{2}\omega_n)$. Let us now consider the action of X_s on $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ sitting in the homogeneity one part and assume it acts trivially. Then due to the previous isomorphism, this kernel is $\text{Ker}(X_s^2)$ when X_s^2 is acting on $L(-\frac{1}{2}\omega_n)$ in homogeneity zero. As before, the commutator $[X_s^2, D_s]$ acts by zero. Because it is equal to $-X_s(\mathbb{E} + n) - (\mathbb{E} + n)X_s$, it acts on homogeneity zero elements by $-(2n+1)X_s$ and due to the fact that X_s is an isomorphism, it is nonzero and so yields the contradiction. In conclusion, $X_s : L(\omega_{n-1} - \frac{3}{2}\omega_n) \rightarrow L(-\frac{1}{2}\omega_n)$ acting between homogeneity one and two is an isomorphism. Clearly, one can iterate the procedure further using the subsequent Lemmas 5.1 and 5.2 on $sp(2n)$ -invariant

intertwining operators acting on the direct sum of simple highest weight $sp(2n)$ -modules. An analogous induction procedure can be used to prove the isomorphic action of D_s .

In what follows, we formulate the previous qualitative statements more quantitatively. Denote by \mathcal{M}_l^+ resp. \mathcal{M}_l^- the (irreducible) simple $sp(2n)$ -module with highest weight $L(l\omega_1 - \frac{1}{2}\omega_n)$ resp. $L(l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n)$, and call them **symplectic monogenics** of degree l (or l -homogeneous symplectic monogenics). We put $\mathcal{M}_l := \mathcal{M}_l^+ \oplus \mathcal{M}_l^-$ and we also have

$$\mathcal{M}_l = \ker D_s \cap (\mathcal{P}_l \otimes \mathcal{C}).$$

We then obtain two auxiliary lemmas.

Lemma 5.1. *Suppose $M_\ell \in \mathcal{M}_\ell$ is a symplectic monogenic of degree ℓ . Then*

$$D_s(X_s^k M_\ell) = \frac{1}{2}k(2n + 2l + k - 1)X_s^{k-1}M_\ell.$$

Proof. By induction. □

Lemma 5.2. *Suppose $M_\ell \in \mathcal{M}_\ell$ is a symplectic monogenic of degree ℓ . Then*

$$D_s^j(X_s^k M_\ell) = c_{j,k,\ell} X_s^{k-j} M_\ell$$

with

$$c_{j,k,\ell} = \begin{cases} \frac{1}{2^j} \frac{k!}{(k-j)!} \frac{(2n + 2l + k - 1)!}{(2n + 2l + k - j - 1)!} & j \leq k \\ 0 & j > k. \end{cases}$$

Proof. The lemma follows from j iterations of Lemma 5.1. □

The previous considerations can be summarized in the symplectic analog of the classical theorem on separation of variables in the orthogonal case, see e.g. [17] and the references therein.

Theorem 5.3. *The space $\mathcal{P} \otimes \mathcal{C}$ decomposes under the action of \mathfrak{sl}_2 into the direct sum of simple highest weight $sp(2n)$ -modules*

$$(5.7) \quad \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j \mathcal{M}_l,$$

where we used the notation $\mathcal{M}_l := \mathcal{M}_l^+ \oplus \mathcal{M}_l^-$. The decomposition takes the form of an infinite triangle

$$\begin{array}{cccccccc}
\mathcal{P}_0 \otimes \mathcal{C} & \mathcal{P}_1 \otimes \mathcal{C} & \mathcal{P}_2 \otimes \mathcal{C} & \mathcal{P}_3 \otimes \mathcal{C} & \mathcal{P}_4 \otimes \mathcal{C} & \mathcal{P}_5 \otimes \mathcal{C} & \cdots & \\
& & & \parallel & & & & \\
\mathcal{M}_0 \longrightarrow & X_s \mathcal{M}_0 \longrightarrow & X_s^2 \mathcal{M}_0 \longrightarrow & X_s^3 \mathcal{M}_0 \longrightarrow & X_s^4 \mathcal{M}_0 \longrightarrow & X_s^5 \mathcal{M}_0 & \cdots & \\
& & & \oplus & & & & \\
& & \mathcal{M}_1 \longrightarrow & X_s \mathcal{M}_1 \longrightarrow & X_s^2 \mathcal{M}_1 \longrightarrow & X_s^3 \mathcal{M}_1 \longrightarrow & X_s^4 \mathcal{M}_1 & \cdots \\
& & & \oplus & & & & \\
& & & \mathcal{M}_2 \longrightarrow & X_s \mathcal{M}_2 \longrightarrow & X_s^2 \mathcal{M}_2 \longrightarrow & X_s^3 \mathcal{M}_2 & \cdots \\
& & & \oplus & & & & \\
& & & \mathcal{M}_3 \longrightarrow & X_s \mathcal{M}_3 \longrightarrow & X_s^2 \mathcal{M}_3 & \cdots & \\
& & & & & & & \\
& & & & & \mathcal{M}_4 \longrightarrow & X_s \mathcal{M}_4 & \cdots \\
& & & & & & & \\
& & & & & & \mathcal{M}_5 & \cdots
\end{array}$$

where all summands are simple highest weight $sp(2n)$ -modules. The k -th column gives the decomposition of homogeneous polynomials of degree k taking values in $\mathcal{C} = L(-\frac{1}{2}\omega_n) \oplus L(\omega_{n-1} - \frac{3}{2}\omega_n)$. The l -th row forms a highest weight \mathfrak{sl}_2 -module $\bigoplus_{j=0}^{\infty} X_s^j \mathcal{M}_l$ generated by the space of symplectic monogenics \mathcal{M}_l .

One immediate Corollary is the structure of polynomial solutions of the symplectic Dirac operator on \mathbb{R}^{2n} . The statement is given for both symplectic spin modules $L(-\frac{1}{2}\omega_n)$ and $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ separately.

Corollary 5.4. *The kernel of (half of) the symplectic Dirac operator D_s acting on $L(-\frac{1}{2}\omega_n)$ -valued polynomials is*

$$Ker^+(D_s) \simeq \bigoplus_{l \in \mathbb{N}} \left(L(2l\omega_1 - \frac{1}{2}\omega_n) \oplus L((2l+1)\omega_1 - \frac{1}{2}\omega_n) \right).$$

The kernel of (half of) the symplectic Dirac operator D_s acting on $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ -valued polynomials is

$$Ker^-(D_s) \simeq \bigoplus_{l \in \mathbb{N}} \left(L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L((2l+1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \right).$$

Every homogeneous polynomial of degree k , taking values in \mathcal{C} , can now be decomposed in monogenic components as follows.

Theorem 5.5. *Let $p \in \mathcal{P}_k \otimes \mathcal{C}$. Then there exists a unique representation of p as*

$$(5.8) \quad p = \sum_{i=0}^k p_i,$$

where $p_i = X_s^{k-i} m_i$ and $m_i \in \mathcal{M}_i$.

We now proceed to construct projection operators that allow to explicitly compute the representation given in Theorem 5.5. They are given in the following theorem.

Theorem 5.6. *The operators*

$$(5.9) \quad \pi_i^k = \sum_{j=0}^{k-i} a_j^{i,k} X_s^{i+j} D_s^{i+j}$$

with

$$a_j^{i,k} = (-1)^j (2n + 2k - 2i - 1) \frac{2^{i+j}}{i!j!} \frac{(2n + 2k - 2i - j - 2)!}{(2n + 2k - i - 1)}$$

and $i = 0, \dots, k$ satisfy

$$\pi_i^k(X_s^j \mathcal{M}_{k-j}) = \delta_{ij} X_s^i \mathcal{M}_{k-i}.$$

Proof. Using lemma 5.2 it is easy to see that $\pi_i^k(X_s^j \mathcal{M}_{k-j}) = 0$ for all $j < i$. The coefficients $a_j^{i,k}$, for fixed i and k , can now be determined iteratively. First of all, expressing $\pi_i^k(X_s^i \mathcal{M}_{k-i}) = X_s^i \mathcal{M}_{k-i}$ yields

$$a_0^{i,k} = \frac{1}{c_{i,i,k-i}} = \frac{2^i (2n + 2k - 2i - 1)!}{i! (2n + 2k - i - 1)!}.$$

Similarly, expressing $\pi_i^k(X_s^{i+1} \mathcal{M}_{k-i-1}) = 0$ then yields

$$a_1^{i,k} = -\frac{c_{i,i+1,k-i-1}}{c_{i+1,i+1,k-i-1}} a_0^{i,k} = -\frac{1}{n + k - i - 1} a_0^{i,k}.$$

Thus continuing we arrive at the hypothesis

$$a_j^{i,k} = (-1)^j \frac{2^j (2n + 2k - 2i - j - 2)!}{j! (2n + 2k - 2i - 2)!} a_0^{i,k}$$

which can be proven using induction. Indeed, suppose that the statement holds for $a_j^{i,k}$, $j \leq l$, then we prove that it also holds for $a_{l+1}^{i,k}$. This last coefficient has to satisfy

$$\sum_{j=0}^{l+1} a_j^{i,k} c_{i+j,i+l+1,k-i-l-1} = 0.$$

Substituting the known expressions we obtain

$$\begin{aligned} a_{l+1}^{i,k} &= -\sum_{j=0}^l a_j^{i,k} \frac{c_{i+j,i+l+1,k-i-l-1}}{c_{i+l+1,i+l+1,k-i-l-1}} \\ &= -\sum_{j=0}^l a_j^{i,k} \frac{2^{l+1-j}}{(l+1-j)!} \frac{(\alpha - 2l - 1)!}{(\alpha - l - j)!} \\ &= -\frac{2^{l+1}}{(l+1)!} \frac{(\alpha - 2l - 1)!}{\alpha!} a_0^{i,k} \sum_{j=0}^l (-1)^j \binom{l+1}{j} \frac{(\alpha - j)!}{(\alpha - l - j)!} \end{aligned}$$

where we have put $\alpha = 2n + 2k - 2i - 2$. The proof is now complete by remarking that

$$\sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} \frac{(\alpha - j)!}{(\alpha - l - j)!} = 0.$$

This can either be obtained directly (see e.g. Lemma 5 in [5]) or as a consequence of Gauss's hypergeometric theorem, expressing ${}_2F_1(a, b; c; 1)$ in terms of a product of Gamma functions. \square

Note that there exists another way of computing the projection operators on irreducible summands, namely using the Casimir operator of \mathfrak{sl}_2 . First observe that

$$\Gamma_s \mathcal{M}_k = -\frac{1}{2}k(2n-1+k)\mathcal{M}_k.$$

It is then clear that the operators

$$\mathbb{P}_i^k = \prod_{j=0, j \neq i}^k \frac{2\Gamma_s + j(2n-1+j)}{j(2n-1+j) - i(2n-1+i)}, \quad i = 0, \dots, k$$

defined on the space $\mathcal{P}_k \otimes \mathcal{C}$ satisfy

$$\mathbb{P}_i^k(X_s^{k-j}\mathcal{M}_j) = \delta_{ij}X_s^{k-i}\mathcal{M}_i.$$

6. OPEN QUESTIONS AND UNRESOLVED PROBLEMS

In [15], the symplectic Dirac operator D_s on \mathbb{R}^{2n} was studied as an $sp(2n+2)$ -invariant differential operator in the context of contact parabolic geometry. As a consequence, the kernel of D_s has the structure of an $sp(2n+2)$ -module. Clearly, $\text{Ker}(D_s)$ is, as a vector space, isomorphic to $\text{Pol}(\mathbb{R}^{2n+2})$ (see Corollary 5.4) and we leave the question of its representation theoretic content open.

In [17], the authors studied the specific deformation of Howe duality and Fischer decomposition for the Dirac operator acting on spinor valued polynomials, coming from the Dunkl deformation of the Dirac operator. It is an interesting question to develop the Dunkl version of the symplectic Dirac operator in the context of symplectic reflection algebras (see [10]).

Another interesting question is whether the reproducing kernel of the space of symplectic monogenics \mathcal{M}_l can again be expressed in terms of Gegenbauer polynomials, as in the orthogonal and Dunkl case.

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