III CONFERENZA

CAUCHY INTEGRAL FORMULA

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There is a lot of function theory developed for regular functions of a quaternionic variable. A part of the theory is described in the quaternionic setting in [Su] , together with further references (there is described e.g. the Cauchy integral formula and the Laurent series expansion). A lot of further results can be found as a special case of more general theory of regular functions on R_n with values in Clifford algebra (e.g. mean value theorem, maximum modulus theorem, Morera theorem, Runge type theorems, Mittag-Leffler theorem, Hilbert modules with reproducing kernel and a lot of transform analysis - see [BDS]).

This lecture will be devoted to one special topic, namely to Cauchy integral formula. It will be shown, how the analogue of the classical Cauchy integral formula from the complex case, can be, after the complexification, transformed into quite different type of integral formulas, used in mathematical physics.

(III.1) Cauchy integral formula.

Let us try to follow again the analogy with the complex case. If f is holomorphic in $\Omega \subset C$, if γ is a cycle in Ω and $p \notin \langle\gamma\rangle$, then

$$2\pi i \mathrm{Ind}_\gamma\, p . f(p) = \int_\gamma f(z)/(z-p)dz$$

It is clear that the integration over a cycle γ has to be substituted in quaternionic analysis by the integration over a 3-dimensional cycle.

The use of differential forms and integration over chains is the best language for the Cauchy integral formula in higher dimension. We shall use here cubical differentiable singular chains (i.e. differentiable maps of the unit interval in R_4 into H), as described in [M] . The integration of quaternionic valued forms is done componentwise, hence the standard Stokes theorem holds for them.

Conferenza tenuta nel mese di Maggio 1982

Now, the complex 1-form dq/q has to be substituted by a quaternionic-valued 3-form. The suitable 3-form can be found by analogy. In the complex case we have:

i) $d(fdz) = 0$ iff $\bar{\partial} f = 0$ iff f is holomorphic

ii) $1/z = \bar{z}/|z|^2 = \partial/\partial z(\log|z|^2)$, where $\log|z|^2$ is the fundamental solution of the Laplace equation in the plane

It is easy to see that the 3-form

$Dq = dx_1 \wedge dx_2 \wedge dx_3 - i_1 dx_0 \wedge dx_2 \wedge dx_3 - i_2 dx_0 \wedge dx_2 \wedge dx_1 - i_3 dx_0 \wedge dx_1 \wedge dx_2$

has the property

i) $d(fDq) = 0$ iff $\partial^\dagger f = 0$ $(f: H \longrightarrow H)$

and because $1/|q|^2$ is the fundamental solution of the Laplace operator in R_4, it suggests to use its derivative

$$\partial(1/|q|^2) = (-2)q^\dagger/|q|^4 = -2/q.1/|q|^2$$

as the kernel.

To rewrite the complex formula, the notion of index is still missing. In the complex case, the cycle $\gamma_1 = \exp(it)$, $t \in \langle 0, 2\pi\rangle$ is the generator of the first homology group of $C \setminus \{0\}$ and $k = \mathrm{Ind}_\gamma 0$ tells us that γ is homological to $k.\gamma_1$.

The same situation can be met in quaternionic analysis, too.

(III.2) Definition.

Let Γ be a 3-cycle in $H \setminus \{p\}$, then we shall define the index of p with respect to Γ by

$$\mathrm{Ind}_\Gamma\, p = 1/2\pi^2 \int_\Gamma Dq(q-p)^\dagger/|q-p|^4$$

Remark:

The function $1/|q|$ being harmonic, it is clear that $q^\dagger/|q|^4$ is regular, because $\partial^\dagger(q^\dagger/|q|^4) = \partial^\dagger\partial(1/|q|^2) = \Delta(1/|q|^2) = 0$. Hence the 3-form $Dq.q^\dagger/|q|^4$ is closed and its integral depends only on the class of homology of the corresponding cycle (in $H \setminus \{0\}$).

(III.3) Theorem:

Let Γ be a 3-cycle in H, $p \notin \langle\Gamma\rangle$. Then:

1) $\mathrm{Ind}_\Gamma\, p \in \mathbb{Z}$.

2) $\mathrm{Ind}_\Gamma\, p$ (as a function of p) is constant on every connected component of $H \setminus \langle\Gamma\rangle$.

Proof:

1) The sphere S_3 is the deformational retract of $H \setminus \{0\}$, hence

$$H_3(H \setminus \{0\}) = H_3(S_3) = \mathbb{Z} .$$

It implies at the same time that S_3 is the generator of $H_3(H \setminus \{0\})$, so for every 3-cycle Γ there exists $k \in \mathbb{Z}$ such that Γ is homological to $k.S_3$ in $H \setminus \{0\}$.

Hence $\mathrm{Ind}_\Gamma\, 0 = \mathrm{Ind}_{k.S_3} 0 = k \in \mathbb{Z}$.

2) The function $\mathrm{Ind}_\Gamma\, p$ has to be constant on every connected component of $H \setminus \{\Gamma\}$ because it is continuous and discrete valued.

Example:

Take the 3-sphere S_3 (with the induced orientation) as a cycle in $H \setminus \{0\}$, then we can compute $\mathrm{Ind}_{S_3} 0$. It can be checked by the direct computation that

$$Dq = q/|q| \;.\; dS,$$

where dS is the standard surface element on the sphere. Hence

$$\mathrm{Ind}_{S_3}\, 0 = 1/2\pi^2 \int_{S_3} q^\dagger/|q|^4 Dq = 1/2\pi^2 \int_{S_3} 1/|q|^3 dS = 1/2\pi^2 \int_{S_3} dS = 1 .$$

(III.4) Theorem (Cauchy):

If $\partial^\dagger g = f\partial^\dagger = 0$ in $\Omega \subset H$ and if a 3-cycle Γ is homologically trivial in Ω , then $\int_\Gamma fDqg = 0$

Proof:

It is sufficient to use the Stokes theorem, because the 3-form $fDqg$ is closed: $d(fDqg) = \omega\,[(f\partial^\dagger).g + f.(\partial^\dagger g)] = 0,$
where ω is the volume form.

(III.5) Theorem (Cauchy integral formula):

Let f be a (left) regular function in $\Omega \subset H$ and let Γ be a 3-cycle, homologically trivial in Ω .

Then for every $p \in \Omega \setminus \langle\Gamma\rangle$ we have

$$\mathrm{Ind}_\Gamma\, p.f(p) = 1/2\pi^2 \int_\Gamma (p-q)^\dagger/|p-q|^4.Dqf(q) .$$

Proof:

Denote $k = \mathrm{Ind}\, p$ and consider a sufficiently small 3-sphere S_3 with the center p and a radius ρ. Then Γ is homological in $\Omega\setminus\{p\}$ with $k.S_3$. It follows from the Cauchy theorem that the integration over Γ in the right hand side can be substituted by the integration over $k.S_3$. The radius ρ can be taken arbitrarily small. Hence the function $f(q)$ can be (in the limit) substituted by $f(p)$. But then we have

$$1/2\pi^2 \int_\Gamma (p-q)^\dagger/|p-q|^4.Dqf(q) = 1/2\pi^2 \int_{kS_3} (p-q)^\dagger/|p-q|^4.Dqf(p) = f(p)\mathrm{Ind}_\Gamma p.$$

(III.6) Elliptic and hyperbolic type integral formulas.

The character of the Cauchy integral formula, described above, is quite similar to the complex case. It is a typical elliptic integral formula. The contour of integration here is 3-dimensional, the values of the function have to be known on the contour of integration and values of the regular function are reconstructed 'inside' of the contour of integration.

This integral formula is valid for solutions of Fueter equation, i.e. for solutions of Weyl equation, considered on the complex Minkowski space and restricted after to the 'Euclidean slice' of it.

The Weyl equation, in its original setting in mathematical physics, is considered on (real) Minkowski space and it is a hyperbolic equation. There are integral formulas for solutions of Weyl equation, but they have quite different - hyperbolic- character. The contour of integration is 2-dimensional here and depends on the point P, where the value is evaluated (it is the intersection of the initial value 3-surface with the null cone of P). Both values of the field and its derivatives on the contour of integration (derivatives in suitable directions) are needed in the formula. (We shall meet an example of such formula in a while.)

These two types of formulas are so different in character, that it is difficult to see any connection between them. But, as the Weyl equation on Minkowski space and the Fueter equation on the Euclidean slice are both restrictions of the Weyl equation on complex Minkowski space, it is conceivable that there could be a form of a Cauchy integral formula in complex quaternionic analysis, which interpolates between these two, quite different, integral formulas. We want to show now that it is really the case.

The basic principle of the procedure was invented by Imaeda ([I]) in the context of the classical electrodynamics. The procedure can be generalized a lot.

In the (real) quaternionic analysis the value of the integral does not change, when the contour of integration Σ_3 is deformed (without crossing the singularity of the expression under the sign of integration). After the complexification, the freedom for the deformation of the contour is enlarged a lot - it can be deformed now everywhere in complex Minkowski space (without crossing the singularity, which is now the 'complex null cone' and has the complex dimension 3).

Now the contour of integration Σ_3 can be deformed into a family of circles with centres in points of the contour of integration Σ_2 of the hyperbolic integral formula. The circles are small enough and go into complex Minkowski space, avoiding so the singularities.

If the circles are placed well (in complex 1-planes for example), the integration over them can be computed with aid of the residue formula. The residua are computed just in points of Σ_2 . The poles are of order 2, hence to compute residua we need values and first derivatives in points of Σ_2. So after 1-dimensional integration being done, we are left just with hyperbolic type integral formula in (real) Minkowski space.

We shall discuss now the transition from elliptic to hyperbolic integral formulas in two cases - for Penrose's integral formula for solutions of the Weyl equation and for Riesz's integral formula for solutions of the wave equation.

(III.7) <u>Cauchy integral formula for solutions of Weyl equation</u>.

<u>Theorem</u>:

Denote $CN_P = \{P+Q \mid Q \in CH , |Q|^2 = 0\}$. Suppose $P \in \Omega \subset CH$ and consider the ball $U_\rho=\{P+Q \mid Q \in H , |Q|^2 \leq \rho^2\}$ and the sphere $S_3 = \partial U_\rho$. Suppose that $\partial^+.F = 0$ in Ω and that Σ_3 is a 3-cycle homological with S_3 in $\Omega \setminus CN_P$. Then

$$F(P) = 1/2\pi^2 \int_{\Sigma_3^-} (Q-P)^+/|Q-P|^4 \, . \, DQ.F(Q) \, .$$

<u>Proof</u>:

It is sufficient to prove that the corresponding 3-form is closed.

But the coefficients of the form are holomorphic, hence the same computation as in the real case gives us the needed information.

Remark.

1. There are 3 other forms of the Cauchy integral formula for the other forms of Fueter equation. We shall use another one - for solutions of the equation $\partial F = 0$ we have

$$F(P) = 1/2\pi^2 \int_{\Sigma_3} \{(Q-P)/|Q-P|^4\} DQ^{\dagger} F(Q)$$

2. The spinor translation of this equation is

$$\Phi_{A'}(P) = 1/2\pi^2 \int_{\Sigma_3} \{(Q_{AA'}-P_{AA'})/|Q-P|^4\} DQ^{AB'} \Phi_{B'}(Q) .$$

(III.8) Penrose's integral formula.

It is an integral formula giving values of a spinor field $\Phi_{A'}$ using initial data on the null cone in Minkowski space. To describe it, we have to use the spinor language. Let us recall that points on the null cone N of the origin in Minkowski space can be expressed as $x_{AA'} = \zeta_A\bar{\zeta}_{A'}$, $\zeta_A \in S_A$. Take a point P inside the forward null cone of origin and denote Σ_2 the intersection of the (backward) null cone N_P of the point P with the initial data null cone N. Take a typical point Q' on Σ_2. Suppose that $\zeta_A \in S_A$ are homogeneous coordinates for $\mathbb{P}_1(\mathbb{C})$ worth of null directions in N_P and suppose that the vector $Q'P$ is described in the spinor language as $r\zeta_A\bar{\zeta}_{A'}$, $r=r(\zeta_A) > 0$. Let us choose further a (smooth) spinor field $\xi_A=\xi_A(Q')$ on the null cone N such that the vector $\xi_A\bar{\xi}_{A'}$ is tangent to the generator of N. We shall suppose the normalization condition $\xi^A\zeta_A = 1$ on Σ_2. Then

$$\Phi_A(P) = 1/2\pi \int_{\Sigma_2} (\zeta_A/r)\{D\theta - (2\rho+\varepsilon)\}\, dS ,$$

where $\theta = \Phi_A\xi^A$, $D = \xi^A\bar{\xi}^{A'}\nabla_{AA'}$, $\rho = -\xi^A\zeta^B\bar{\xi}^{B'}\nabla_{BB'}\xi_A$, $\varepsilon = -\zeta^A D\xi_A$ and dS is the surface element of Σ_2.

We want to show now that the complexified Cauchy integral formula reduces to Penrose's formula on M after one integration being done. Suppose that the field Φ_A satisfies the equation $\nabla^{AA'}\Phi_A = 0$ in a neighbourhood $U \subset CM$ of the set $\{z_{AA'}=P_{AA'}-t\zeta_A\bar{\zeta}_{A'} | \zeta_A \in C_2, t\in\langle 0,r\rangle\} \subset N$. The contour $\Sigma_3 = \{Q_{AA'}-r\zeta_A\bar{\zeta}_{A'}+z\xi_A\bar{\xi}_{A'} \,|\, \zeta_A \in C_2,\ z \in C_2,\ |z| = \rho\}$

belongs to the same class of homology as the small sphere S_3 around P (it can be proved either by a direct homotopy argument (see [So3]), or by computing the index of Σ_3 as in [BFMS]). Hence the Cauchy integral from complex quaternionic analysis can be used for the contour Σ_3 and it is only question of (a long) computation to see that the integration with respect to z can be carried out using the residue formula and that the singularity of the expression under the integral sign lies just on Σ_2. At the end of computation the Penrose's integral formula is recovered. For details see [So3].

(III.9) An integral formula for complex Laplacian.

An integral formula for complex Laplacian in $\mathbb{C}_n$ was described in [B] . We shall describe it shortly in the quaternionic case $n=3$. The expression under the integral sign is a little bit more complicated, the formula is another typical example of an elliptic type integral formulas.

Theorem:

Take $P \in \Omega \subset \mathbb{C}\mathbb{H}$ and Σ_3, S_3 same as in (III.7).
If a mapping $F : \mathbb{C}\mathbb{H} \longrightarrow \mathbb{C}\mathbb{H}$ satisfies $\partial\partial^{\dagger}F = 0$, then

$$F(P)=1/4\pi^2\int_{S_3}\{(Q-P)^{\dagger}/|Q-P|^4\}DQF(Q) + \tfrac{1}{2}D^{\dagger}Q\partial^{\dagger}F/|Q-P|^2$$

Remark.

The equation and the formula are both 'reducible' in the sense that both are acting componentwise. Hence the attention can be restricted to functions with values in $\mathbb{C} \subset \mathbb{C}\mathbb{H}$.

Proof (main idea):

It is easy to compute that the form under the integral sign is closed. After the transition to the small sphere around P, the value F(P) can be substituted instead of F(Q) and the rest of the integral will be equal to one again.

(III.10) Riesz's integral formula.

An integral formula for the wave equation in 'Minkowski' space with n spacial dimension was described by Riesz in [R] . We shal describe it again for the case $n=3$.

Let N be the (backward) null cone of the origin and let Σ_3 be a 3-dimensional space-like surface in Minkowski space M. Denote Σ_2

the intersection of S_3 with N. We can suppose that the surface Σ_2 has the parametrization $b = b(\lambda_1,\lambda_2)$; λ_1,λ_2 being in a 2-dimensional domain of parameters. Consider a vector function $c(\lambda_1,\lambda_2)$ on Σ_2, satisfying:

i) $(c,c) = 0$, i.e. the vector c is null vector
ii) $(c,db) = 0$, i.e. c is normal to the tangent planes of Σ_2
iii) $(c,b) = \frac{1}{2}$.

Denote

$$D(\tau,\lambda) = \det \begin{bmatrix} b^0 & c^0 & \frac{\partial b^0}{\partial\lambda_1}+\tau\frac{\partial c^0}{\partial\lambda_1} & \frac{\partial b^0}{\partial\lambda_2}+\tau\frac{\partial c^0}{\partial\lambda_2} \\ b^1 & c^1 & . & . \\ b^2 & c^2 & . & . \\ b^3 & c^3 & . & . \end{bmatrix}$$

and

$$F(\tau,\lambda) = \frac{D(\tau,\lambda)}{D(0,\lambda)} .$$

Then we have

<u>Theorem</u>.

Let u be a solution of the wave equation in a neighbourhood of the set $\{tb(\lambda_1,\lambda_2) \mid t\in\langle 0,1\rangle;\ \lambda_1,\ \lambda_2 \text{ as for } \Sigma_2\}$. Then

$$u(0) = -1/\pi \int_{\Sigma_2} \left(\tfrac{1}{2}\frac{\partial F}{\partial\tau}\, u + F\,\frac{\partial u}{\partial\tau}\right)\Big|_{\tau=0} dS .$$

(III.11) <u>The transition from (III.9) to (III.10)</u>.

The 3-cycle $\Sigma_3 = b(\lambda_1,\lambda_2)+\tau c(\lambda_1,\lambda_2)$ λ_1,λ_2 as above, $\tau = \rho$, τ $C\}$. satisfies the assumptions of Theorem (III.9), hence the corresponding integral formula can be used and after the integration with respect to τ being done, the formula (III.10) is recovered. More details can be found in [B].

(III.12) <u>Remarks</u>.

1. The complex quaternionic Cauchy integral formula (and its generalization to complex Clifford case) is very useful and versatil tool. The classical Kirchhoff formulas for the wave equation and for Maxwell field could be derived in a similar way. Moreover, there are the integral formulas for the description of solutions of massless field equation by homogeneous functions of a twistor variable (see [ATT]). The complexified Cauchy integral formula can be used also for the description of the inverse twistor correspondence (see [So2]).

2. Even if a considerable amount of results was created in the hypercomplex analysis, there is clearly an outstanding problem to be solved, namely to extend the theory to a suitable type of manifolds. The principal difficulty lies in the fact that regular functions have some peculiar properties, different from holomorphic functions. The product and the composition of two regular functions need not be regular again. So the standard definition of the complex manifold cannot be generalized to the quaternionic case. A lot of work and some new ideas are needed to solve this problem. A step in the right direction was done in [M] .

(III.13) Aknowledgement.

It's my pleasure to thank prof. A.V.Ferreira for the invitation to come to Bologna and for a lot of useful discussions. I have found in Bologna pleasent and nice working conditions and I surely want to thank for them to prof. Ferreira as well as to prof. S.Coen and drs. M. Idà and M.Manaresi. A part of results mentioned in the lectures was finished during my stay in Bologna.

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